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Worthington, W. R., M.A.; Southport.
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ERRATA.

P. 74, l. 3. *For* "be" *read* "is."
P. 107, last line. *For* "its" *read* "their."

ADDENDUM.

P. 51. Question 15703.—Professor Sanjána, M.A., directs attention to *Reprint*, Vol. LXX. [Original Series], pp. 116–7.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

15636. (C. E. YOUNGMAN, M.A.)—Given two circles and a point, draw two parallel tangents, one to each circle, equidistant from the point.

Solutions (I.) *by* R. F. DAVIS, M.A.; (II.) *by* GEORGE SCOTT, M.A.; (III.) *by* JAMES BLAIKIE, M.A.

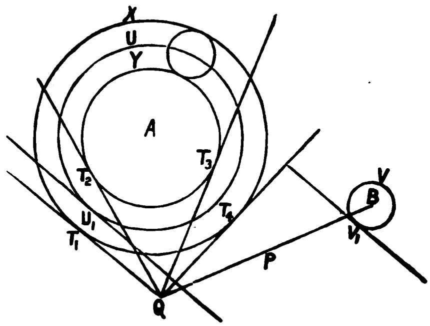
(I.) The envelope of a straight line parallel to, and equidistant from, two parallel tangents (one to each of two given circles) is a circle having its centre at the middle point of the join of the centres of the given circles and its radius equal to the semi-sum or semi-difference of their radii.

The solution of this problem therefore reduces to the simple operation of drawing tangents from a given point to two determinate concentric circles; so that there may be either four solutions, or two, or none.

(II.) Given two circles and a fixed point, draw parallel tangents, one to each, equally distant from the fixed point. Suppose the problem done and a parallel to the tangents drawn through the given point. If the points of contact are on opposite sides of the line joining the centres, the difference of the perpendiculars from the centres on the parallel through the given point will be equal to the difference of the radii, and is therefore given. Hence bisect the line joining the centres, join the bisection to the given point, and on this line describe a semicircle; with the bisection as centre and half the given difference as radius describe a circle; where this cuts the semicircle join to the given point: this joining line gives the direction of the required tangents. The problem thus reduces to a very old one, namely: Given three points, to draw through one of them a line so that the difference of the perpendiculars drawn to it from the other two may be given. Suppose that the points of contact are on the same side of the line joining the centres. Then the perpendicular from one centre on the parallel through the given point will be radius + half

the interval of tangents, while the perpendicular on the same line falling from the other centre will be radius − half the interval; therefore the sum of the perpendiculars will equal the sum of the radii, and is therefore given. Accordingly proceed as before, but use half the sum instead of half the difference of the radii as the distance at which the semicircle is to be cut.

(III.) Let A be the centre of circle U, B of circle V, and let the radius of U be greater than that of V. Let P be the given point. Join BP



and produce it its own length to Q. With A as centre and radii equal to the sum and difference of the circles, describe the circles X, Y. From Q draw tangents QT_1, QT_2, QT_3, QT_4 to these circles. To any of these four tangents parallel tangents can be drawn to the circles U and V such that P is equally distant from them. This follows at once from the fact that P is equidistant from Q and B.

[The Proposer remarks that it is clear from these solutions (Mr. Blaikie's especially) that the construction might be given thus: Draw a circle B′ symmetrical to B with respect to P, and then the common tangents to A and B′, and lastly parallels to these touching B. And the same first step will serve when the problem is to draw two equal circles touching each other at P; one of them to touch A also, and the other to touch B.]

15591. (Lt.-Col. Allan Cunningham, R.E.)—Excluding $m = X^4+1$, state the forms of m for which $X^{12}+1 \equiv 0 \pmod{m}$. Show how to find X when m is given. *Ex.*, $m = 99961$.

Solution by the Proposer.

m must be either a prime $p = 24\varpi+1$, or a product of such primes. A root of $x^{12}+1 \equiv 0 \pmod{m}$ may be found by resolving m into its 2-ic partitions $$m = a^2+b^2 = c^2+2d^2 = A^2+3B^2.$$

Then, if x_3, x_4, x_{12} be (proper) roots of the congruences, $x^3-1 \equiv 0$, $x^4+1 \equiv 0$, $x^{12}+1 \equiv 0 \pmod{m}$, it is known (see page xvii of the Intro-

duction to the present writer's large *Tables of Quadratic Partitions*, now in course of publication) that x_3, x_4, x_{12} may be found from the linear congruences $x_3 \equiv (A-B)/2B$, $x_4 \equiv (c/2d).(a-b)/b$, $x_{12} \equiv x_3 . x_4 \pmod{m}$.

Ex.—$m = 99961 = p$. Here

$$m = 295^2+156^2 = 293^2+2.84^2 = 277^2+3.88^2.$$

Then $x_3 \equiv (277-88)/2.88 \equiv 189/176$, whence $176x_3 \equiv 189 \pmod{p}$. Solving, this gives $x_3 = 15336$. Next,

$$x_4 \equiv \frac{293}{2.84}.\frac{275-156}{156} \equiv \frac{293}{168}.\frac{119}{156} \equiv \frac{4981}{32.9.13},$$

whence $32.9.13x_4 \equiv 4981 \pmod{p}$. Solving, this gives $x_4 = 6062$.

Lastly, $x_{12} \equiv 15336 . 6062 \pmod{p}$, whence $x_{12} = \pm 3102$. The other six roots ($< p$) may be formed as the least positive or negative residues of $x_{12}^5, x_{12}^7, x_{12}^{11} \pmod{p}$.

Note.—The *Tables of Quadratic Partitions* above quoted give the above partitions of all primes not greater than 100,000 (if capable thereof), and therefore enable the roots of the following congruences to be found:— $x^2+1 \equiv 0$, $x^3 \mp 1 \equiv 0$, $x^4+1 \equiv 0$, $x^6+1 \equiv 0$, $x^{12}+1 \equiv 0 \pmod{p}$ for all such primes up to the limit 100,000 by the solution of linear congruences only (see page xvii above quoted).

15485. (Professor Nanson.)—If t_r be the arithmetic mean of the r-th powers of n positive quantities which are not all equal, prove that $t_1, t_2^{\frac{1}{2}}, t_3^{\frac{1}{3}}, t_4^{\frac{1}{4}}, \ldots$ are in ascending order of magnitude.

Additional Solution by the Proposer.

Compounding the arrays

$$\begin{matrix} a^r, & b^r, & \ldots, & & a, & b, & \ldots, \\ a^{r-1}, & b^{r-1}, & \ldots, & & 1, & 1, & \ldots, \end{matrix}$$

we get
$$s_{r+1}s_{r-1}-s_r^2 = \Sigma a^{r-1}b^{r-1}(a-b)^2,$$

where $s_r = \Sigma a^r$. Hence, if $a, b, \ldots$ are positive and not all equal, we have $s_r^2 < s_{r+1}s_{r-1}$, and hence also $t_r^2 < t_{r+1}t_{r-1}$. Thus we have

$$t_1 < t_2/t_1 < t_3/t_2 < \ldots < t_r/t_{r-1} < t_{r+1}/t_r;$$

whence, by multiplication, $(t_{r+1}/t_r)^r > t_r$; so that $t_r^{1/r} < t_{r+1}^{1/(r+1)}$, and hence $t_1 < t_2^{\frac{1}{2}} < t_3^{\frac{1}{3}} < \ldots < t_r^{1/r}$.

15601. (Professor Neuberg.)—Soient A′ le milieu du côté BC d'un triangle ABC; B′ et B″ les points où AC est rencontré par la bissectrice intérieure et la bissectrice extérieure de l'angle B; C′ et C″ les points de

rencontre de AB avec les bissectrices intérieure et extérieure de l'angle C. Les circonférences A'B'C', A'B''C'' peuvent-elles toucher le côté BC sans que l'on ait AB = AC?

Solution by the PROPOSER *and* W. F. BEARD, M.A.

1. Si la circonférence A'B'C' touche BC et coupe encore AB en C_1 et AC en B_1, on a

$$(BA')^2 = BC'.BC_1, \quad (CA')^2 = CB'.CB_1, \quad AC'.AC_1 = AB'.AB_1,$$

$$BC' = ac/(a+b), \quad AC' = bc/(a+b), \quad CB' = ab/(a+c), \quad AB' = bc/(a+c);$$

donc

$$BC_1 = \frac{a(a+b)}{4c}, \qquad AC_1 = \pm(c - BC_1) = \pm\left[c - \frac{a(a+b)}{4c}\right],$$

$$CB_1 = \frac{a(a+c)}{4b}, \qquad AB_1 = \pm\left[b - \frac{a(a+c)}{4b}\right].$$

Le signe $\pm$ s'applique respectivement au cas où les points C_1, B_1 sont situés entre A et BC ou au-delà de A ; c'est le même signe pour les deux points. La condition $AC'.AC_1 = AB'.AB_1$ se traduit par l'égalité

$$\left[c - \frac{a(a+b)}{4c}\right]\frac{bc}{a+b} - \left[b - \frac{a(a+c)}{4b}\right]\frac{bc}{a+c} = 0,$$

qu'on ramène facilement à

$$(b-c)\left[4bc(a+b+c) + a(a+b)(a+c)\right] = 0.$$

Elle exige $b - c = 0$.

2. Si la circonférence A'B''C'' touche BC et coupe encore AB en C_2, AC en B_2, il faut que C'' et B'' soient situés du même côté de BC, par exemple du côté opposé à A. La condition $AC''.AC_2 = AB''.AB_2$

donne alors $$\left[c + \frac{a(b-a)}{4c}\right]\frac{bc}{b-a} - \left[b + \frac{a(c-a)}{4b}\right]\frac{bc}{c-a} = 0,$$

ou $$(b-c)\left[a^3 - a^2(b+c) + 5abc - 4bc(b+c)\right] = 0.$$

Cette équation convient encore au cas où C'' et B'' sont situés au-delà de A par rapport à BC. Elle est vérifiée si $b - c = 0$ ou si

$$f(a) \equiv a^3 - a^2(b+c) + 5abc - 4bc(b+c) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Supposons $b > c$. Comme $f(b) = -4bc^2 < 0$, $f(b+c) = bc(b+c) > 0$, l'équation (1) admet au moins une valeur de a comprise entre $b-c$ et $b+c$. D'ailleurs, l'équation (1) a une seule racine entre b et $(b+c)$. En effet, les racines de l'équation $f'(a) = 0$ sont

$$a' = \tfrac{1}{3}\left[b + c - \surd(b^2 - 13bc + c^2)\right], \quad a'' = \tfrac{1}{3}\left[b + c + \surd(b^2 - 13bc + c^2)\right].$$

Si le rapport b/c est compris entre $\frac{1}{2}(13 - \surd 165)$ et $\frac{1}{2}(13 + \surd 165)$, a' et a'' sont imaginaires, la dérivée $f'(a)$ est toujours positive et la fonction $f(a)$ croît constamment avec a et ne peut s'annuler qu'une seule fois. Si $b/c < \frac{1}{2}(13 - \surd 165)$ ou $> \frac{1}{2}(13 + \surd 165)$, a' et a'' sont réelles et l'équation $f(a) = 0$ a au plus une racine réelle dans chacun des intervalles $(-\infty, a')$, (a', a''), (a'', ∞). Donc l'intervalle $(b, b+c)$, étant compris dans l'intervalle (a'', ∞), a une seule racine de $f(a) = 0$.

15620. (D. BIDDLE.)—ABC being a given plane triangle of which I is the in-centre, draw tangents to the in-circle across the angles, so that the three resulting triangles may be equal and have a maximum area.

Solution by W. F. BEARD, M.A.

Let x, y, z be the lengths of the tangents which cut off the required equal triangles of maximum area. Then the areas of the triangles are $r(s-a-x)$, $r(s-b-y)$, $r(s-c-z)$. Thus $a+x = b+y = c+z$, and these are to be of minimum value. If $a > b > c$, then the least value of x is the length of the tangent which is bisected by the arc nearest to A; if this length $= a_1$, then* $y = a-b+a_1$, $z = a-c+a_1$. Drawing tangents of these lengths, we obtain the required triangles.

[The PROPOSER suggests the following method for completing the construction of the required tangents:—x, y, z being the bases of the three equal triangles, let h_x, h_y, h_z be the respective heights. Then $xh_x = yh_y = zh_z$, and, h_x being now given, it is easy to geometrically determine h_y, h_z from the values found of x, y, z. Next, join BI, CI, and on them, as diameters, describe circles; also draw from B, C chords equal respectively to h_y+r, h_z+r joining the distant extremities with I. The required intercepts will be afforded by parallels (to these joins) tangential to the in-circle of the given triangle, the points of contact being found by parallels through I to the respective chords.]

15668. (Professor NANSON.)—The locus of the meet of perpendicular planes through two fixed lines is a quadric. Show that the three quadrics thus derived from the three pairs of opposite edges of a tetrahedron have a common curve of intersection.

Solution by W. H. BLYTHE, M.A.

Let a, b, c, d be the tetrahedron. Draw ae perpendicular to bcd; then the plane aec is perpendicular to the base plane bcd, so that ce is one of the generators of the conicoid through the opposite edges ac, bd. Therefore the point e is on the conicoid, and may be similarly shown to be on the other two conicoids. We find then that the four points in the faces of the tetrahedron where the perpendiculars from the opposite vertices meet them are on the three conicoids; so also are the angular points of the tetrahedron.

Since the conicoids have eight points in common, they have a common curve of intersection. (Salmon's *Geometry of Three Dimensions*, p. 110.)

If we take A, B, C, D as the areas of the faces, and represent the

* N.B. $a-b+a_1 < s-b$, for $a_1 < s-a$, which shows that it is possible to draw y, z of the required lengths.

angle between the planes A, B by (AB), using areal co-ordinates, the equation to the curve becomes

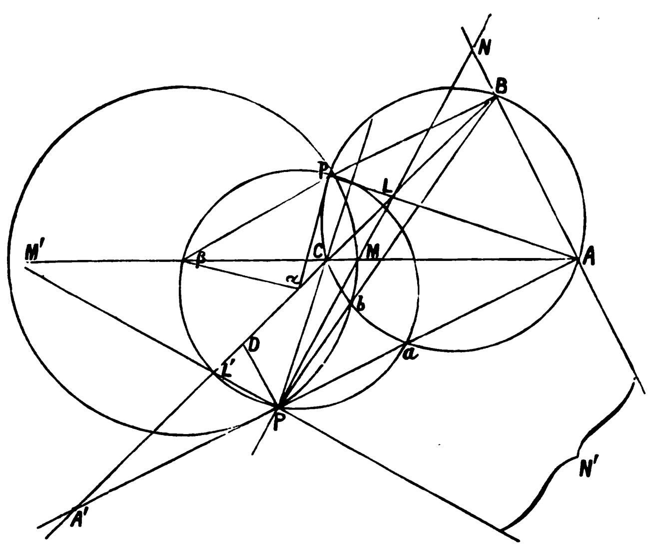
$$\mathrm{BC}\cos(\mathrm{BC})\,\alpha\delta + \mathrm{AD}\cos(\mathrm{AD})\,\beta\gamma = \mathrm{DC}\cos(\mathrm{DC})\,\beta\alpha + \mathrm{AB}\cos(\mathrm{AB})\,\gamma\delta$$
$$= \mathrm{BD}\cos(\mathrm{BD})\,\alpha\gamma + \mathrm{AC}\cos \mathrm{AC}\,\beta\delta.$$

Note.—This problem may also be proved by projective axial pencils.

15687. (C. E. Hillyer, M.A. Suggested by Question 14111, Vol. lxxi.)—Prove that the mid-points of the segments intercepted by the axes of a conic on the sides of a self-conjugate triangle are collinear; and that the circles described on these segments as diameters cointersect on the circum-circle of the triangle.

Solution by the Proposer.

Let ABC be a triangle self-conjugate with respect to a conic whose centre is P, and let the axes meet BC in L and L′. Join PA, meeting the circum-circle of ABC in a, and BC produced in A′. Draw PD parallel to AB to meet BC in D. Then PA′ and the parallel to BC through P are conjugate radii, as also are PD, PC and PL′, PL; therefore L, C, D, L′ form an involution of which A′ is the centre; therefore



A′C . A′D = A′L . A′L′. But, since CDP = supplement of ABC = CaA, P, D, C, a are concyclic; therefore A′C . A′D = A′a . A′P; and therefore A′L . A′L′ = A′a . A′P; therefore L, a, P, L′ are concyclic, *i.e.*, the

circle on LL′ passes through a. Similarly the circle on MM′ passes through b, and the circle on NN′ through c.

It is required then to prove that the circles with their centres on the sides of the triangle ABC and passing through P, a and P, b and P, c respectively cointersect on the circum-circle. Let the first of these circles whose centre is α, the mid-point of LL′, meet the circum-circle in p, and let β be the centre of the circle $\mathrm{P}bp$; it will be sufficient to prove that β is in AC.

Now $\quad p\mathrm{AB} = pb\mathrm{B} =$ supplement of $pb\mathrm{P} = \frac{1}{2}p\beta\mathrm{P} = \mathrm{P}\beta\alpha$

and $\quad p\mathrm{BA} = p\alpha\mathrm{P} = p\alpha\beta$;

therefore the triangles $p\mathrm{BA}$ and $p\alpha\beta$ are directly similar; and therefore also the triangles $\mathrm{A}p\beta$ and $\mathrm{B}p\alpha$ are directly similar; therefore the angle between $\beta\mathrm{A}$ and $\alpha\mathrm{B}$ is equal to $\mathrm{A}p\mathrm{B}$; therefore $\beta\mathrm{A}$ meets $\alpha\mathrm{B}$ on the circum-circle, *i.e.*, at C; therefore β is in CA. Similarly, the third circle also passes through p. Thus the three circles cointersect on the circum-circle and their centres are collinear.

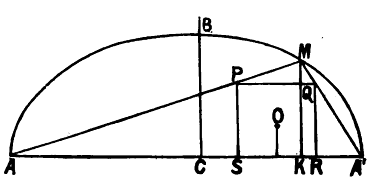
Note.—In general one pair of perpendiculars can be drawn through a given point so as to intercept segments on the sides of a triangle whose mid-points are collinear, viz., the axes of the conic having the given point as centre with respect to which the triangle is self-conjugate.

If, however, the given point is the orthocentre of the triangle, then the conic is a circle (real or imaginary) of which any two diameters at right angles are conjugate. Hence Question 14111 is seen to be a particular case of the above.

[The Editor regrets that the diagram is imperfect, and that correction is impossible without entire reproduction.]

15619. (Professor Neuberg.)—Soient A, A′ les extrémités d'un axe d'une ellipse, M un point mobile sur cette courbe. On inscrit au triangle MAA′ un carré PQRS dont le côté RS repose sur AA′. Trouver les lieux des points P, Q et du centre du carré.

Solutions (I.) *by* Professor Sanjana, M.A.; (II.) *by* R. Tucker, M.A., *and others.*



(I.) Drawing the ordinate MK, we have

$$\mathrm{PS}/\mathrm{SA} = \mathrm{MK}/\mathrm{KA},$$

$$\mathrm{PS}/\mathrm{RA}' = \mathrm{QR}/\mathrm{RA}' = \mathrm{MK}/\mathrm{KA}';$$

hence $\mathrm{PS}^2/(\mathrm{SA}\,.\,\mathrm{RA}') = \mathrm{MK}^2/(\mathrm{AK}\,.\,\mathrm{KA}') = \mathrm{BC}^2/\mathrm{AC}^2$.

Let $\mathrm{PS} = y$, $\mathrm{AS} = x$; then

$$y^2/[x(2a-x-y)] = b^2/a^2 \quad \text{or} \quad a^2y^2+b^2x^2+b^2xy-2ab^2x = 0,$$

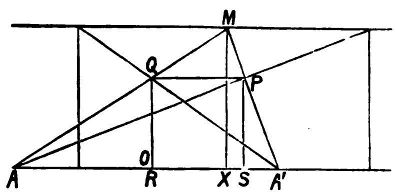
an ellipse going through A. So the locus of Q will be an ellipse going through A′.

If O be the centre of the square, with co-ordinates f, g, we have $f = x + \frac{1}{2}y$, $g = \frac{1}{2}y$, so that $y = 2g$, $x = f - g$; hence, for O,

$$a^2 . 4g^2 + b^2(f-g)^2 + b^2 . 2g(f-g) - 2ab^2(f-g) = 0,$$

i.e., $$b^2f^2 + (4a^2 - b^2)g^2 - 2ab^2f + 2ab^2g = 0,$$

and the locus is an ellipse going through both A and A′.



(II.) Let the perpendicular $\text{MX} = p$; side of square $= y$. Then $\text{OX}^2/a^2 + p^2/\beta^2 = 1$ (i.), O, the midpoint of AA′, being the origin.

Now $p/y = 2a/(2a-y)$...(ii.).

$\text{OX} = a - \text{XA}'$

$= a - (a-x)(p/y)$

$= [a(2x-y)]/(2a-y)$ (iii.).

Hence $$[(2x-y)/(2a-y)]^2 + 4a^2y^2/[\beta^2(2a-y)^2] = 1$$

by (i.), i.e., $$\beta^2(2x-y)^2 + 4a^2y^2 = \beta^2(2a-y)^2,$$

i.e., $$\beta^2x^2 - \beta^2xy + a^2y^2 = \beta^2(a^2 - ay),$$

i.e., an ellipse, because $\beta^2 < 4a^2$ is the locus of P.

Again, for locus of Q, if $\text{OR} = x'$, then $\text{OX} = [a(2x'+y)]/(2a-y)$, and the locus is $\beta^2x^2 + \beta^2xy + a^2y^2 = \beta^2(a^2 - ay)$, and the result is similar to the former.

Again, if X′, Y′ be the co-ordinates of the centre of the square in one position, $x' + x = 2\text{X}' = 2x' + y = 2x' + 2\text{Y}'$ and $y = 2\text{Y}'$; hence $x' = \text{X}' - \text{Y}'$.

Substituting for x in the last equation, we get

$$\beta^2(\text{X}'-\text{Y}')^2 + 2\beta^2\text{Y}'(\text{X}'-\text{Y}') + 4a^2\text{Y}'^2 = \beta^2(a^2 - 2a\text{Y}')$$

or $$\beta^2\text{X}'^2 + (4a^2 - \beta^2)\text{Y}'^2 = \beta^2(a^2 - 2a\text{Y}');$$

therefore the locus is an ellipse, because $2a > \beta$.

15671. (Professor Sanjána, M.A.)—From the centre of curvature at every point of a central conic the two normals other than the radius of curvature are drawn. Prove that the envelope of the chord joining the feet of these normals is $x^{\frac{2}{3}}/a^{\frac{2}{3}} \pm y^{\frac{2}{3}}/b^{\frac{2}{3}} = 1$, and that the locus of the pole of this chord is $a^2/x^2 \pm b^2/y^2 = 1$, the conic being referred to its axes.

Solution by A. S. Tombe, M.A., *and* M. V. A. Sastry, B.A.

The feet of the four normals passing through (α, β) lie on the conic

$$c^2xy + b^2\beta x - a^2\alpha y = 0 \quad \text{.........................} \quad (1).$$

If $lx + my + 1 = 0$ join one pair of feet and $l'x + m'y + 1 = 0$ the other pair, then

$$(lx + my + 1)(l'x + m'y + 1) + x^2/a^2 + y^2/b^2 - 1 = 0 \quad \text{.... ......} (2)$$

is a conic through four feet and also through the centre of the ellipse.

Comparing (1) with (2), $ll' = -1/a^2$, $mm' = -1/b^2$ If $lx+my+1=0$ touches the ellipse, then

$$1 = a^2l^2+b^2m^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

we have therefore to find the envelope of

$$l'x+m'y+1=0 \quad\text{or}\quad x/a^2l+y/b^2m-1=0 \quad\ldots\ldots\ldots\ldots(4)$$

with condition (3).

Differentiating in the usual way, we get

$$a^2l/(a^2y-a^2b^2m) = b^2m/(b^2x-a^2b^2l) = 1/-a^2b^2lm.$$

Using (3), we get $x = a^4l^3$, $y = b^4m^3$; therefore $(x/a)^{\frac{2}{3}}+(y/)^{\frac{2}{3}} = 1$. For the locus of the pole of (4) compare it with $\mathrm{X}x/a^2+\mathrm{Y}y/b^2 = 1$; therefore $x = 1/l$, $y = 1/m$; therefore, using (3), $a^2/x^2+b^2/y^2 = 1$. For the hyperbola change b^2 into $-b^2$.

15579. (*Communicated by* Rev. T. Wiggins, B.A.)—Sum to infinity the series $\sin^2\theta - 3/2\sin^2 2\theta + 3^2/3\sin^2 3\theta - 3^3/4\sin^2 4\theta + \ldots$.

[*Note.*—This was one of several series proposed for solution at an examination conducted by the Board of Education. Is the series convergent?]

Solution by C. M. Ross.

In the first place let

$$\begin{aligned}
\mathrm{S} &= \sin^2\theta - \tfrac{1}{2}m\sin^2 2\theta + \tfrac{1}{3}m^2\sin^2 3\theta - \tfrac{1}{4}m^3\sin^2 4\theta \ldots \\
&= \tfrac{1}{2}(1-\tfrac{1}{2}m+\tfrac{1}{3}m^2-\tfrac{1}{4}m^3+\ldots) - \tfrac{1}{2}(\cos 2\theta - \tfrac{1}{2}m\cos 4\theta + \tfrac{1}{3}m^2\cos 6\theta - \ldots) \\
&= (1/2m)\log(1+m) - (1/2m)\log(1+2m\cos 2\theta+m^2) \quad [\theta \neq (2n+1)\tfrac{1}{2}\pi].
\end{aligned}$$

Now $m \not> 1$. Hence the given series is not convergent. If m were $\frac{1}{3}$ instead of 3,

$$\mathrm{S} = \tfrac{3}{2}\log\tfrac{4}{3} - \tfrac{3}{2}\log(\tfrac{4}{3}+\tfrac{2}{3}\cos 2\theta) = \tfrac{3}{2}\log 2 - \tfrac{3}{2}\log(2+\cos 2\theta).$$

15668. (Lt.-Col. Allan Cunningham, R.E.)—Find a general expression for triangular numbers consisting of a certain digit repeated n times followed by another (different) digit repeated n times.

Solution by the Proposer.

A slight examination of triangular numbers will show that

$$x = 666\ldots6 = 6\,.\,\tfrac{1}{9}(10^n-1)$$

is the only number giving $\mathrm{T} = \frac{1}{2}x(x+1)$ of form required for small values of n. And in the general case

$$\begin{aligned}
&\tfrac{1}{2}[6\,.\,\tfrac{1}{9}(10^n-1)][6\,.\,\tfrac{1}{9}(10^n-1)+1] = \tfrac{1}{9}(10^n-1)(2\,.\,10^n+1) \\
&= [2\,.\,\tfrac{1}{9}(10^n-1)]\,10^n + \tfrac{1}{9}(10^n-1) = 222\ldots22111\ldots111,
\end{aligned}$$

(n two's followed by n units), as required.

Algebraical Note on proving the Formula for ${}_nH_r$.

By A. M. Nesbitt, M.A.

The method usually given is complicated and difficult to remember. I venture to suggest the following as an improvement.

We have to classify, or rather arrange, r things into n compartments. Draw $n-1$ strokes marked 1, 2, 3, ..., $n-1$

$$a \underset{1}{|} b \underset{2}{|} c \underset{3}{|} d \ldots \underset{n-1}{|} n;$$

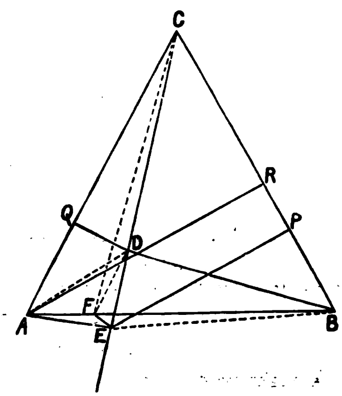
this will give us the n compartments marked $a, b, c, \ldots, n$. In these compartments distribute r dots in any fashion; thus, for ${}_4H_7$,

$$\underset{a}{\vdots\!:} \;|\; \underset{b}{} \;|\; \underset{c}{:} \;|\; \underset{d}{}$$

the arrangement given would signify a^5c^2. The problem is then seen to be that of arranging $n+r-1$ things, of which $n-1$ are strokes and r are dots. But this is plainly $(n+r-1)!/(n-1)!\,r!$ or $n(n+1)\ldots(n+r-1)/r!$.

15475. (George Scott, M.A.)—(Corollary to Question 15272.) In an *isosceles* triangle a line is drawn through the vertex C, and perpendiculars are dropped on it from A and B, meeting it in E and D respectively. Draw DQ perpendicular to AC, and EP perpendicular to BC. The sum, or the difference, of EP and DQ will be constant according as CE cuts the base or the base produced.

Solution by the Proposer.



If EF is drawn parallel to BC, ED (by the solutions) will be parallel to AC. Therefore

$$\Delta BEC = \Delta BFC,$$

while $\Delta ADC = \Delta AFC$.

Therefore

$$BC(EP + DQ) = BC \,.\, AR,$$

AR being perpendicular to BC; therefore

$$EP + DQ = AR.$$

15553. (Professor Nanson.)—Eliminate x, y, z from

$$ax^2 + by^2 + cz^2 = 0, \quad (b-c)\,lyz + (c-a)\,mzx + (a-b)\,nxy = 0,$$
$$lx + my + nz = 0.$$

Additional Solution by the PROPOSER.

The equations are all satisfied if $ax : by : cz = l : m : n$, provided the single relation $$bcl^2 + cam^2 + abn^2 = 0 \quad \text{.............................} \quad (1)$$
holds, and they are also all satisfied if
$$x : y : z = \sqrt{(b-c)} : \sqrt{(c-a)} : \sqrt{(a-b)},$$
provided the single relation
$$l\sqrt{(b-c)} + m\sqrt{(c-a)} + n\sqrt{(a-b)} = 0 \quad \text{.................} \quad (2)$$
holds. Rationalizing (2) and multiplying by (1), we get the result of elimination in the form given by Dr. Muir.

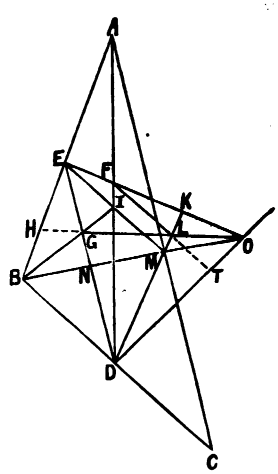
Otherwise, the eliminant of any two ternary quadrics and a linear is $4\Delta\Delta' - \Phi^2$ where Δ, Δ' are the bordered discriminants of the quadrics and Φ is the intermediate of Δ, Δ'. Now in the present case we have
$$\Phi = 2lmn\Sigma a(b-c) = 0;$$
so that the eliminant is $4\Delta\Delta'$ where Δ is the first member of (1) and Δ' is the first member of the rationalized form of (2).

Problem with Original Solution.

Communicated by S. C. BAGCHI.

AB and BC are two adjoining sides of a regular polygon inscribed in a circle. D is the middle point of BC. Show by methods of pure geometry that the straight line joining the centre of gravity of the triangle ABD to the centre of the inscribing circle is perpendicular to AD.

Bisect AB at E. Draw EO and DO perpendicular to AB and BC respectively. Join AC, ED. Join BO, intersecting ED and AC at N and M respectively. Join EM, intersecting AD at I. Join BI, intersecting ED at G. Join OG and produce it to H in AB. Join DM and produce it to intersect OH at L and OE at K. Join FL and let it intersect OD at T.



G is evidently the centre of gravity of the triangle ABD. The figure EBDM is a rhombus. The angle DKF is a right angle. Consider the triangle EMK and the transversal FID.

$$KF . EI . MD = FE . IM . KD.$$

But EI = IM ; therefore

$$KF/FE = KD/MD.$$

Consider the triangle EBN and the transversal HGO.

$$EH . BO . NG = HB . NO . GE.$$

But GE = 2NG ; therefore EH/HB = 2 (NO/BO).

Now $KD/MD = 1+MK/MD$. Let $MK = d$, $DM = a$, $NM = c$, $MO = b$. $KD/MD = 1+d/a = 1+b/(b+2c) = 2\,(ON/BO)$; therefore

$$KF/FE = EH/BH = KL/LM$$

(since KM is parallel to EB); therefore FL is parallel to EM; the angle FTD is a right angle. Hence L is the orthocentre of the triangle FDO; therefore DA is perpendicular to OG.

10261. (Professor Deprez.)—On décrit des coniques semblables entre elles, ayant même centre O et passant par un même point P. Démontrer que (1) les sommets et les foyers de ces courbes décrivent des podaires de coniques; (2) le centre du circle osculateur en P décrit une quartique; (3) les directrices enveloppent une conique.

Solution by Frances E. Cave.

(1) Let A be the vertex of one of the conics; describe that conic of the series for which P is the extremity of the minor axis; let K be its vertex, OL its semi-diameter perpendicular to OA. Then, since $\angle POA = \angle KOL$, $OP : OA :: OL : OK$. Therefore A lies on the pedal of the fixed conic PKL. The foci and feet of directrices lie on pedals of similar and similarly situated conics. Hence the directrices envelop a conic.

(2) Take P as the origin, PO as the axis of x; let (x, y) be the centre of the osculating circle, OD the semi-diameter conjugate to OP. Then, with the usual notation, $\rho = OD^2/p$, $x/\rho = p/OP$; therefore

$$OD^2 = p\rho = x\,.\,OP\,;$$

therefore $$\frac{CA^2+CB^2}{CA\,.\,CB} = \frac{OD^2+OP^2}{pOD} = \frac{OP\,(OP+x)}{\sqrt{(xOP)}\,OP\,(x/\rho)}.$$

But $(CA^2+CB^2)/(CA\,.\,CB)$ is constant, $= k$, say; therefore the required locus is the quartic $(x+OP)^2\,(x^2+y^2) = k\,OP\,x^3$.

15646. (R. W. D. Christie.)—Solve the equation $X^2-19Y^2 = -3$ (in integers) by the use of other convergents than the ordinary Pellian, and prove its generality: thus $p_n/q_n = \frac{4}{1}, \frac{5}{1}, \frac{17}{4}, \frac{22}{5}, \frac{61}{14}$ (ad inf.).

Solutions (I.) *by the* Proposer; (II.) *by* A. H. Bell.

(I.) Let $P_n^2-\pi Q_n^2 = R_n$ and $p_n^2-\pi q_n^2 = r_n$, where P, Q are ordinary convergents, R_n ordinary residues, p, q the new convergents and r_n the new residues, i the integer of the square root of the prime

π, m_n the common multipliers for both sets. Then, by Euler's theorem extended, we have

$$(P_n^2-\pi Q_n^2)(p_n^2-\pi q_n^2) = A^2-\pi B^2 = R_n r_n.$$

Now, for $R_n = 1$, which is always possible by the ordinary theory, $A^2-\pi B^2$ *must* conform with $p_n^2-\pi q_n^2$; but for the new convergents we can put $-\frac{i}{1}$, $\frac{1(\pi-i^2)}{0}$, $\frac{m_1(\pi-i^2)-i}{1}$, $\frac{m_2 m_1(\pi-i^2)-m_2 i+\pi-i^2}{m_2}$, ..., $\frac{Q_n(p-i^2)-Q_{n-1}i}{q_n}$. Thus

$$\{Q_n(\pi-i^2)-iQ_{n-1}\}^2-\pi q_n^2 = P_n^2-\pi Q_n^2$$

for the penultimate convergents. Thus

$$p_n = Q_n(\pi-i^2)-iq_n,$$

where $$\frac{p_n}{q^n} = \frac{3}{0}, \frac{2}{1}, \frac{5}{1}, \frac{17}{4}, \frac{22}{5}, \frac{61}{14}.$$

$\pi = 19$; *e.g.:*

$$\begin{aligned} 61 &= 39.3-4.14, \\ 22 &= 14.3-4.5, \\ 17 &= 11.3-4.4, \\ 5 &= 3.3-4.1, \\ 2 &= 2.3-4.1, \\ 3 &= 1.3-4.0. \end{aligned}$$

For the sake of comparison, I take two primes 19 and 43.

$\frac{P_n}{Q_n} = \frac{4}{1}, \frac{9}{2}, \frac{13}{3}, \frac{48}{11}, \frac{61}{14}, \frac{170}{39}, \ldots, \frac{1421}{326}, \ldots$, ordinary convergents;

$\frac{p_n}{q_n} = \frac{4}{1}, \frac{3}{0}, \frac{2}{1}, \frac{5}{1}, \frac{17}{4}, \frac{22}{5}, \frac{61}{14}, \ldots, \frac{170}{39}, \frac{401}{92}, \ldots$, new ,,

Again, for the prime 43, we have

$\frac{P_n}{Q_n} = \frac{6}{1}, \frac{7}{1}, \frac{13}{2}, \frac{46}{7}, \frac{59}{9}, \frac{341}{52}, \frac{400}{61}, \frac{1541}{235}, \frac{1941}{296}, \frac{3482}{531}, \ldots$, old;

$\frac{p_n}{q_n} = -\frac{6}{1}, \frac{7}{0}, \frac{1}{1}, \frac{8}{1}, \frac{25}{4}, \frac{33}{5}, \frac{190}{29}, \frac{223}{34}, \frac{859}{131}, \frac{1082}{165}, \frac{1941}{296}, \ldots$, new.

It will thus be seen the penultimate convergents of the ordinary correspond with the ultimate convergents of the new, and also other laws correspond, *e.g.:* $59.9 = 531$ in the old with $33.5 = 165$ new, &c.; see previous Vols.

It is clear that similar series for the ante-penultimates can be formed for other values of the dexter, and interesting deductions drawn.

(II.) The negative development of $\sqrt{19}$ by continued fractions gives for convergents $\frac{1}{0}, \frac{-5}{1}, \frac{-4}{1}, \frac{-13}{3}, \frac{61}{-14}, \ldots$. Since $\frac{-4}{1}$ and $\frac{61}{-14}$ precede -3, the denominator of the complete quotient, they represent x/y of this equation.

Again, by the method employed of solving $X^2-NY^2 = \pm 1$ without the aid of continued fractions (found in a former number of *The Educational Times*), the triple series and its development will answer every possible

condition as given by both the positive and negative development in the $\sqrt{19}$, by continued fractions.

$$1^2\times 19 = 4^2+3 = 5^2-6,\quad 2^2\times 19 = 9^2-5,\quad \dots\,.$$

From this we form the triple series

Terms	diff.	0,	1,	2 & diff. 1,	2,	3,	4,	...		
$X = n$	$= 9$	$= 5$	$+4$,	13,	22,	35,	48,	61,	...	
$Y = m$	$= 2$	$= 1$	$+1$,	3,	5,	8,	11,	14,	...	
$D = d$	-5,	-6,	$+3$,	$+2$,	-9,	-9,	-5,	$+3$,	...	

developed by two differences

$$D = +9,\ -1,\ -11,\ -0,\ +4,\ +8,\ \dots;$$
$$D_2 = -10,\ -10,\ 11,\ +4,\ +4,\ \dots\,.$$

Every column solves $X^2-19Y^2 = \pm D$ by changing the sign of d.

[Lt.-Col. ALLAN CUNNINGHAM, R.E., discusses the given equation as follows:—The solution of the proposed equation does not require the use of convergents if the known solutions of the unit form $\tau^2-19v^2 = 1$ be admitted. These may be found by taking the powers of the fundamental form $170^2-19.39^2 = 1$. The successive results are (see the writer's recently published *Tables of Quadratic Partitions*)

$$\tau = 170,\ 57799,\ 19651490,\ \&c.;\quad v = 39,\ 13260,\ 4508361,\ \&c.$$

Calling any of these (τ_n, v_n), and noting that $4^2-19.1^2 = -3$ is the minimum solution of the given equation, all the other solutions (X_n, Y_n) may be obtained from the formulæ $X_n = 4\tau_n \mp 19v_n$, $Y_n = 4v_n \mp \tau_n$ (using the same sign in each of X_n, Y_n).

The successive results are

$$X = 4;\ 61,\ 1421;\ 20744,\ 483136;\ \&c.$$
$$Y = 1;\ 14,\ 326;\ 4759,\ 110839;\ \&c.$$

15665. (A. M. NESBITT, M.A.)—(1) If $x_1, x_2, \dots, x_n$ be the roots of the equation $\quad p_0x^n - p_1x^{n-1} + p_2x^{n-2} - \dots + (-1)^n p_n = 0$,

prove that the product of the $n(n-1)/(1.2)$ quantities of which x_r+x_s is the type (r not being equal to s) may be written as a determinant of order $n-1$ whose κ-th row is $p_{2-\kappa}, p_{4-\kappa}, \dots, p_{2(n-1)-\kappa}$ with the convention that $p_a = 0$ if $a < 0$ or $> n$.

Example: If $n = 4$, the product is $\begin{vmatrix} p_1 & p_3 & \cdot \\ p_0 & p_2 & p_4 \\ \cdot & p_1 & p_3 \end{vmatrix}$, and, if $n = 5$, it is

$$\begin{vmatrix} p_1 & p_3 & p_5 & \cdot \\ p_0 & p_2 & p_4 & \cdot \\ \cdot & p_1 & p_3 & p_5 \\ \cdot & p_0 & p_2 & p_4 \end{vmatrix}.$$

It will be noticed that the principal diagonal is $p_1, p_2, \dots, p_{n-1}$.

Solution by the PROPOSER.

If the equation $p_0x^n - p_1x^{n-1} + \dots + (-1)^n p_n = 0$ have two roots whose sum is zero (say α and $-\alpha$), it will be true if the sign of α is changed. Hence, at once, $p_0\alpha^n + p_2\alpha^{n-2} + \dots = 0$, $p_1\alpha^{n-1} + p_3\alpha^{n-3} + \dots = 0$; whence, eliminating dialytically,

$$\begin{vmatrix} p_1 & p_3 & p_5 & \dots & & \dots & \\ p_0 & p_2 & p_4 & \dots & & \dots & \\ \cdot & p_1 & p_3 & \dots & & \dots & \\ \cdot & p_0 & p_2 & \dots & & \dots & \\ & \dots & & \dots & p_{n-2} & p_n & \\ & \dots & & \dots & p_{n-3} & p_{n-1} & \end{vmatrix} = 0.$$

This determinant consequently vanishes when the sum of any two roots vanishes, and is therefore divisible by all such expressions as $x_r + x_s$, these being any two different roots; but it is of the degree $1+2+\dots+n-1$, *i.e.*, $\frac{1}{2}[n(n-1)]$, and consequently can only have a numerical factor; while, as it contains the term $x_1^{n-1}x_2^{n-2}\dots x_n$ in the principal diagonal, which is plainly a term in the continued product, this numerical factor is $+1$. Thus the determinant is equal to the product.

15675. (R. TUCKER, M.A.)—ABC is a triangle and O_1, (O_1') are the centres of the circles $B\Omega C$, $(B\Omega' C)$ respectively. Similar points are taken for the other angles of the triangle. Prove that

$$\Sigma (O_1A)^2 + \Sigma (O_1'A)^2 = \tfrac{1}{4}k (\operatorname{cosec}^2 \omega + 8) - 3R^2 \quad (k = \Sigma a^2).$$

Solutions (I.) *by* Professor SANJÁNA, M.A.; (II.) *by the* PROPOSER.

(I.) The angle $B\Omega C = \pi - C$; therefore $\angle BO_1C = 2C$. Let M_1 be the mid-point of BC; then $\angle CO_1M = C$. On O_1M draw AD perpendicular; then $MD = P_1$. Thus

$$O_1A^2 = O_1M^2 + MA^2 + 2O_1M \,.\, MD = \tfrac{1}{4}a^2\cot^2 C + m_1^2 + p_1 a \cot C$$
$$= \tfrac{1}{4}a^2\cot^2 C + m_1^2 + 2\Delta \cot C.$$

Similarly $\qquad O_1'A^2 = \tfrac{1}{4}a^2\cot^2 B + m_1^2 + 2\Delta\cot B.$

Hence $\qquad \Sigma O_1A^2 + \Sigma O_1'A^2$

$$= \tfrac{1}{4}\Sigma a^2(\cot^2 B + \cot^2 C) + 2\Sigma m^2 + 2\Delta\Sigma(\cot B + \cot C)$$
$$= \tfrac{1}{4}\Sigma\left[a^2(\cot^2 A + \cot^2 B + \cot^2 C) - a^2\cot^2 A\right]$$
$$+ \tfrac{1}{2}\Sigma 4m^2 + 4\Delta(\cot A + \cot B + \cot C)$$
$$= \tfrac{1}{4}(\cot^2 A + \cot^2 B + \cot^2 C)(a^2+b^2+c^2) - \tfrac{1}{4}\Sigma a^2\cot^2 A + \tfrac{3}{2}\Sigma a^2 + 4\Delta\cot\omega$$
$$= \tfrac{1}{4}(a^2+b^2+c^2)\left[(\cot A + \cot B + \cot C)^2\right.$$
$$\left. - 2(\cot A\cot B + \cot B\cot C + \cot C\cot A)\right] - \Sigma R^2\cos^2 A + \tfrac{3}{2}\Sigma a^2 + \Sigma a^2$$
$$= \tfrac{1}{4}(a^2+b^2+c^2)(\cot^2\omega - 2) + \tfrac{5}{2}(a^2+b^2+c^2) - \Sigma R^2\cos^2 A$$
$$= \tfrac{1}{4}(a^2+b^2+c^2)(\cot^2\omega + 8) - \Sigma R^2\cos^2 A$$
$$= \tfrac{1}{4}(a^2+b^2+c^2)(\operatorname{cosec}^2\omega + 8) - \tfrac{1}{4}(a^2+b^2+c^2) - \Sigma R^2\cos^2 A$$
$$= \tfrac{1}{4}(a^2+b^2+c^2)(\operatorname{cosec}^2\omega + 8) - \Sigma R^2\sin^2 A - \Sigma R^2\cos^2 A$$
$$= \tfrac{1}{4}(a^2+b^2+c^2)(\operatorname{cosec}^2\omega + 8) - 3R^2.$$

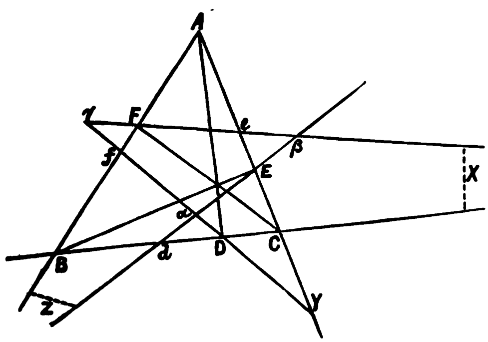
(II.) The circle $B\Omega C$ touches AC at A, and therefore the radius (O_1C) is $\frac{1}{2}a$ cosec C. Hence $O_1A^2 = O_1C^2 + AC^2 = \frac{1}{4}a^2 \operatorname{cosec}^2 C + b^2$. Analogously $O_1'A^2 = \frac{1}{4}a^2 \operatorname{cosec}^2 B + c^2$. Hence

$$\Sigma (O_1A^2 + O_1'A^2) = R^2\Sigma (a^4b^2 + a^2b^4)/a^2b^2c^2 + 2k$$
$$= \ldots = \tfrac{1}{4}k (\operatorname{cosec}^2 \omega + 8) - 3R^2.$$

15682. (R. Tucker, M.A.)—ABC is a triangle; d, e, f are the mid-points of the sides, and D, E, F the feet of the perpendiculars on the sides BC, CA, AB. De, Df; Ef, Ed; Fe, Fd are joined, and produced, if necessary, thus forming two isoscelian triangles, viz., positive $\alpha\beta\gamma$, negative $\alpha'\beta'\gamma'$. Prove that these triangles and ABC have a common centre of perspective.

Solution by Professor Sanjána, M.A.

Since e bisects AC, and AFC is a right angle, therefore $eF = eA$; $\angle eFA = \angle eAF = A$; thus $\angle CeX = \pi - 2A$, and $\angle eCx = \pi - C$, so that



$\angle eXC = A - B$. Hence

$$CX = \tfrac{1}{2}b\,[\sin 2A/\sin (A-B)] = [b \sin A \cos A/\sin (A-B)];$$

and $\qquad BX = a \cos B \sin A/\sin (A-B)$;

therefore $\qquad BX/XC = (\sin A \cos B)/(\sin B \cos A)$.

So also $\qquad CY/YA = (\sin B \cos C)/(\sin C \cos B)$,

and $\qquad AZ/ZB = (\sin C \cos A)/(\sin A \cos C)$;

so that $BX . CY . AZ = XC . YA . ZB$, and X, Y, Z are collinear. The corresponding sides of ABC and $\alpha\beta\gamma$ thus meeting on a right line, it follows that the lines joining corresponding vertices are concurrent, and the triangles are in perspective. And because these triangles are in perspective, and $\gamma\alpha$, BC meet in D, CA, $\beta\gamma$ meet in e, and so for other pairs of sides; therefore the triangle formed by De, Ef, Fd is in perspective with each of these triangles, and the three have a common centre of perspective. *Vide* Lachlan, § 167.

15648. (C. E. YOUNGMAN, M.A.)—The number 666 is apparently the greatest triangular number which has all its figures alike. Is there proof of this?

Solutions (I.) *by* Professor ESCOTT; (II.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.

(I.) $\frac{1}{2}x(x+1) = a + 10a + 100a + \ldots + 10^{r-1}a = \frac{1}{9}a(10^r - 1)$.

Multiplying by 8 and adding 1,

$$(2x+1)^2 = \tfrac{1}{9}[8a(10^r - 1) + 9] \quad (a = 1, 2, \ldots, 9).$$

We see at once that the values $a = 2, 4, 7, 8, 9$ are impossible, and that 3 is impossible excepting for $r = 1$. This leaves the following cases:—

$$a = 1, \quad 8.10^r + 1 = y^2; \qquad a = 5, \quad 40.10^r - 31 = y^2;$$
$$a = 6, \quad 48.10^r - 39 = y^2.$$

We can determine whether or not any one of these equations has a solution for r below any desired limit, by using Gauss's "method of exclusion." For example, when $a = 5$,

$$y^2 = 40.10^r - 31 \equiv 4(1 + 3^{r+1}) \pmod 7.$$

Therefore, since 4 is a quadratic residue of 7, $1 + 3^{r+1}$ must also be congruent to a quadratic residue, *i.e.*,

$$1 + 3^{r+1} \equiv 0, 1, 2, 4 \pmod 7 \quad \text{or} \quad 3^r \equiv 2, 0, 5, 1 \pmod 7;$$

therefore $\qquad r = 6t + 2, 0, 5.$

Similarly, by taking residues with respect to modulus 13, we get $r = 3s + 2, 0$; and, from modulus (17),

$$r \equiv 0, 2, 3, 5, 10, 11, 12, 14, 15 \pmod{16}.$$

By continuing this process, we can find as many expressions for the form of r as we please. I find that the only value of $r < 122$ is 2. Therefore, there is no triangular number containing less than 123 digits all of whose digits are 5's excepting 55.

In this way I find that there are no triangular numbers of less than 30 digits of the required form excepting 1, 3, 55, 6, 66, 666.

(II.) Goncourt's Table of Triangular Numbers (1762) shows that there are no such numbers less than 200,000,000 consisting of a repeated digit except 55, 66, 666. The condition to be fulfilled is

$$\tfrac{1}{2}x(x+1) = T_x = a \,.\, \tfrac{1}{9}(10^n - 1),$$

where x is the root of the triangular (T_x), a is the repeated digit, and n the number of repetitions. This requires

$$8a \,.\, \tfrac{1}{9}(10^n - 1) + 1 = (2x+1)^2 = \Omega^2 \text{ (an odd square).}$$

Now triangulars can only end in 0, 1, 3, 5, 6, 8; and $a = 0$ gives no result; hence $a =$ one of 1, 3, 5, 6, 8. But $a = 3$ gives $\Omega^2 = 266\ldots6665$, $a = 8$ gives $\Omega^2 = 711\ldots1105$, which have endings impossible for squares; thus $a = 3$ or 8 is inadmissible. Also $a = 1$ gives $\Omega^2 = 888\ldots8889$, $a = 5$ gives $\Omega^2 = 444\ldots4441$, $a = 6$ gives $\Omega^2 = 533\ldots3329$. On extracting the square roots, supplying as many 8's, 4's, or 3's as necessary according as $a = 1, 5$, or 6, two cases occur according as the number of figures in Ω^2 is odd or even. It will be found on actual trial that Ω must

be greater than 10^{10} (when $T_x > 666$); so that T_x must be very large, if possible at all. One of the above square roots is clearly impossible, viz., $a = 5$, with an even number of figures in Ω^2, gives $\Omega = 666 \ldots$ *ad. inf.*; it seems difficult to prove whether the other five square roots are possible or not.

15669. (*Communicated by* A. V. Kutti Krishna Menon, B.A.)—O and O′ are two fixed points, P any point in a curve defined by the equation $1/r - 1/r' = 1/c$ where $r = \text{OP}$, $r' = \text{O'P}$, and c is constant. Prove that the distance between P and the consecutive curve obtained by changing c to $c+\delta c$ is ultimately $\delta c/\sqrt{[1+3c^2/(rr')+a^2c^4/(r^3r'^3)]}$, where $a = \text{OO}'$.

[*Note.*—The Proposer desires to obtain an elegant solution of the above "Smith's Prize" Question.]

Solutions (I.) *by* R. F. Davis, M.A., *and* M. V. A. Sastry, B.A.; (II.) *by* S. C. Bagchi, B.A.

(I.) Since $r'-r = rr'/c$,

$$r^2 - 2rr' + r'^2 = r^2r'^2/c^2$$

and $\cos \text{P} = 1 + rr'/2c^2 - a^2/2rr'$.

Differentiating,

$$(1/r^2)(dr/ds) = (1/r'^2)(dr'/ds),$$

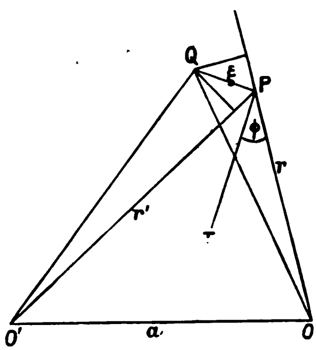
or $\quad r'^2 \cos\phi = r^2\cos(\text{P}-\phi)$;

whence

$$\tan\phi = (r'^2 - r^2\cos \text{P})/r^2 \sin \text{P}$$

and $\cos\phi = r^2 \sin \text{P}/\text{T}$, where

$$\begin{aligned}\text{T}^2 &= r^4 + r'^4 - 2r^2r'^2\cos \text{P}\\ &= 4r^2r'^2(1 + rr'/c^2 + r^2r'^2/4c^4)\\ &\quad -2r^2r'^2\\ &\quad -2r^2r'^2(1 + rr'/2c^2 - a^2/2rr')\\ &= r^4r'^4/c^4 + 3r^3r'^3/c^2 + a^2rr'.\end{aligned}$$



Thus $\text{T} = (r^2r'^2/c^2)\sqrt{(1+3c^2/rr' + a^2c^4/r^3r'^3)}$. Now, taking a point Q on the normal at P such that $\text{PQ} = \xi$, then

$$1/r - 1/r' = 1/c \quad \text{and} \quad 1/(r+\xi\sin\phi) - 1/[r' - \xi\sin(\text{P}-\phi)] = 1/(c+\delta c).$$

Therefore $\qquad \xi\sin\phi/r^2 + \xi\sin(\text{P}-\phi)/r'^2 = \delta c/c^2$,

or $\qquad \xi\,[\sin\phi\,(r'^2/r^2) + \sin(\text{P}-\phi)] = \delta c\,(r'^2/c^2)$,

$$\xi\,[\cos(\text{P}-\phi)\sec\phi\sin\phi + \sin(\text{P}-\phi)] = \ldots, \quad \xi\sin \text{P}\sec\phi = \ldots.$$

But $\cos\phi = r^2\sin \text{P}/\text{T}$. Hence $\xi\text{T} = \delta c\,(r^2r'^2/c^2)$, which gives the required result.

(II.) Consider the potential V at a point P due to charges $(+1, -1)$ placed at O, O′ respectively. Let

$$c\text{V} = 1, \quad (c+\delta c)\,\text{V}_1 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i., ii.)},$$

where c, $c+\delta c$ are constants and δc is very small. Then (i.) and (ii.) are two consecutive equipotential curves. If the electric intensity at P be

R, and if the element of normal to (i.) cut off by (ii.) is $dn = PP'$, we have Rdn = work done when a unit positive charge is taken from P to $P' = -\delta V = \delta c/c^2$ ultimately; therefore

$$dn = \delta c/Rc^2 \quad \text{.................................(iii.)}.$$

But the force along $OP = -\partial V/\partial r$ and that along $O'P = -\partial V/\partial r'$, where $V = 1/r - 1/r' = 1/c$, and $\cos OPO' = (r^2 + r'^2 - a^2)/2rr'$; therefore

$$R^2 = 1/r^4 + 1/r'^4 - (r^2 + r'^2 - a^2)/r^3r'^3 = 1/c^4 + (3/rr')(1/c^2) + a^2/r^3r'^3 \text{...(iv.)}.$$

Substituting in (iii.) the value of R given by (iv.), we get

$$dn = \delta c/\sqrt{(1 + 3c^2/rr' + a^2c^4/r^3r'^3)}.$$

15616. (Professor Nanson)—Four lines 1, 2, 3, 4 determine three quadrics S_1, S_2, S_3, S_4, each passing through three of the lines. Show that S_1, S_2, S_3, S_4 are connected by an identical equation $\Sigma \lambda_{pq} S_p S_q = 0$ where λ_{pq} is a constant which vanishes when the lines p, q intersect.

Solution by Rev. J. Cullen.

Using vectors, a given line p can always be put in the form

$$\rho = a_p \beta_p^{-1} + x\beta_p$$

with
$$S a_p \beta_p = 0 \quad \text{..(1)}.$$

The two lines p and q intersect if ρ is common, which implies

$$S(a_p\beta_q + a_q\beta_p) = 0 \quad \text{..............................(2)}.$$

Put $\mu_p = a_p + V\beta_p\rho$; then, by (1),

$$S\beta_p\mu_p = 0 \quad \text{....................................(3)}$$

and
$$\lambda_{p,q} = S(\beta_q\mu_p + \beta_p\mu_q) \quad \text{.... ....(4)}.$$

Now the quadric determined by the lines p, q, r is

$$S\mu_p\mu_q\mu_r = 0 \quad \text{.................................(5)};$$

so, substituting in $\Sigma\lambda_{p,q}S_pS_q$, we see that this expression takes the form

$$\Sigma S_1 S\beta_1 (\mu_2 S\mu_3\mu_4\mu_1 + \mu_3 S\mu_4\mu_2\mu_1 + \mu_4 S\mu_2\mu_3\mu_1).$$

Observing that in general $\delta S\alpha\beta\gamma = \Sigma\alpha S\beta\gamma\delta$ and that $S_1 = S\mu_4\mu_3\mu_2$, we find
$$\Sigma\lambda_{p,q} S_p S_q = -\Sigma S_p^2 S\beta_p\mu_p.$$

Each term on the right vanishes in virtue of (3); hence the result.

15678. (Lt.-Col. Allan Cunningham, R.E.)—Show that $F_m^4 + F_n^2$, where $F_x = 2^{2^x} + 1$ (a Fermat's number), can always be resolved into *two* factors when $n - m \not< 2$. Write down the co-factors when $n - m = 2$.

Solution by the PROPOSER.

Let $E_x = 2^{2^x}$, so that $E_{x+1} = E_x^2$ and $F_x = E_x + 1$. Let $G_x = E_x - 1$, then $G_x = (E_{x-1}^2 - 1) = (E_{x-1}+1)(E_{x-1}-1) = F_{x-1} . G_{x-1}$. Hence also $G_x = F_{x-1} . F_{x-2} \dots F_2 . F_1 . F_0$. Let $N = F_m^4 + F_n^2$; then

$$N = (F_m^4 + 4) + (F_n^2 - 4).$$

$$\begin{aligned} \text{But}\quad F_m^4 + 4 &= \{(F_m - 1)^2 + 1^2\} . \{(F_m + 1)^2 + 1^2\} \\ &= (E_m^2 + 1) . \{(E_m + 2)^2 + 1\} \\ &= (E_{m+1} + 1) . (E_m^2 + 4E_m + 5) = F_{m+1} . (E_m^2 + 4E_m + 5) \end{aligned}$$

$$\begin{aligned} \text{and}\quad F_n^2 - 4 &= (F_n - 2) . (F_n + 2) = G_n . (E_n + 3) = F_{n-1} . G_{n-1}(E_n + 3) \\ &= (F_{n-1} . F_{n-2} \dots F_2 . F_1 . F_0) . (E_n + 3). \end{aligned}$$

Hence $(F_m^4 + 4)$, $(F_n^2 - 4)$ will both contain F_{m+1} if the latter = any of the factors of G_m, *i.e.*, if $m + 1 \not> n - 1$, or $n - m \not< 2$; and in this case N will will also contain F_{m+1} and will thus be resolvable into (at least) two factors.

Next, if $n - m = 2$, then $F_{m+1} = F_{n-1}$; therefore

$$\begin{aligned} N &= F_{m+1} . \{(E_m^2 + 4E_m + 5) + G_{n-1} . (E_n + 3)\} \\ &= F_{m+1} . \{(E_m^2 + 4E_m + 5) + (E_{m+1} - 1)(E_{m+2} + 3)\} \\ &= F_{m+1} . \{E_m^6 - E_m^4 + E_m^2 + 4E_m + 2\}. \end{aligned}$$

8747. (Professor HAUGHTON, F.R.S.)—The law of cooling of the Sun is $dT/dt = aT^3 - bT$. Integrate this equation, and show the relation between Sun heat and time.

Solution by ERNEST MCKENZIE.

$$dT/dt = aT^3 - bT, \quad dT/dt + bT = aT^3, \quad (1/T^3)(dT/dt) + b/T^2 = a,$$

$$[-\tfrac{1}{2}d(1/T^2)]/dt + b/T^2 = a.$$

Put $\qquad 1/T^2 = z. \quad dz/dt - 2bz = -2a,$

$$ze^{\int -2bdt} = \int -2ae^{\int -2bdt} dt + C, \quad ze^{-2bt} = \int -2ae^{-2bt} dt + C,$$

$$ze^{-2bt} = (a/b)e^{-2bt} + C, \quad z = a/b + Ce^{2bt},$$

$$1/T^2 = a/b + Ce^{2bt} = \text{relation between T and } t.$$

Note.—The EDITOR remarks that, assuming the Sun heat T_0 at a time t_0 known, then the arbitrary constant C may be eliminated, and the required relation obtained in the form

$$(1/T^2 - a/b)/(1/T_0^2 - a/b) = e^{2b(t-t_0)};$$

or, simplifying, $\qquad T^2e^{2bt}/(b - aT^2) = T_0^2 e^{2bt_0}/(b - aT_0^2).$

15528. (A. M. NESBITT, M.A.)—A man, who has m hats of his own in his hall, is visited by n friends, each wearing a hat. They leave their hats with those of their host. When they are going away they are un-

fortunately not in a condition to distinguish between one hat and another. Find the chance that no guest takes away his own hat.

Note by the PROPOSER.

I am loth to disagree with two of our valued contributors, who have independently arrived at the same result as the solution of this problem. It is easy to show that they are wrong; it is much harder to point out where and how they have gone astray.

If we put $m = 0$, we get the old question of letters and envelopes, the result of which is known to be $1/2! - 1/3! + \ldots \pm 1/n!$. Now, if in the result given in the *Reprint*, New Series, Vol. VI., p. 113, we put $m = 0$, we get $(n-1)^n/n^n$, which is therefore manifestly wrong. The method by which I solved the question has already been sufficiently indicated in the Algebraical Note of mine which appears in *The Educational Times*, November, 1904, and in the *Reprint*, New Series, Vol. VII., p. 93. But I can show by two different methods what the chance really is that the first two men to leave take wrong hats. For simplicity I will take $m = 0$, though there is no difficulty in arguing the matter when $m \neq 0$.

First method.—A. may take B.'s hat—chance $1/n$; in this case B. is sure to go wrong. Thus the chance that A. takes B.'s hat and B. *does not take his own* is $1/n$. Again, A. may take a wrong hat (though not B.'s)—chance $(n-2)/n$; in this case B. has also $n-2$ wrong hats to choose from, and the chance he selects a wrong one is $(n-2)/(n-1)$. Thus the chance that A. and B. take wrong hats is

$$1/n + (n-2)^2/[n(n-1)] \quad \text{or} \quad (n^2-3n+3)/[n(n-1)].$$

Second method.—There is one way of their going wrong in which they simply exchange hats: put this aside for the present. Then A. can choose $n-1$ wrong hats; and B. (who is debarred, for the present, from choosing A.'s) may choose $n-2$ wrong hats. Thus the total number of ways in which A. and B. may choose wrong hats is $(n-1)(n-2)+1$, and the chance that they do choose wrong hats is, as before,

$$(n^2-3n+3)/[n(n-1)].$$

I have not the leisure just now to devote to the matter; but I should be very glad to get from some of our contributors a definite reason why the solutions published are wrong and where the two solvers have made a mistake. I have a vague idea; its vagueness is precisely what I want to get rid of. So far as I can see, however, the question cannot be solved by any method analogous to that employed; for we do not seem to be any nearer to a solution when we have discovered the chance that the first two or, for that matter, the first r guests have chosen wrong hats.

10872. (Professor HUDSON, M.A.)—A paraboloid of revolution floats with the lowest point of its base in the surface of a fluid, and its axis inclined at an angle θ to the horizon. Find its height and specific gravity.

Solution by FRANCES E. CAVE.

Let h be the height, $y^2+z^2 = 4ax$ the equation of the paraboloid, $z+\sqrt{(4ah)}+(x-h)\tan\theta = 0$ the equation of the plane of flotation, V the centre of section, and PV the corresponding diameter. The centre of gravity of the solid is $(\frac{2}{3}h, 0, 0)$, and the centre of buoyancy is $[\frac{2}{3}h-\frac{4}{3}\sqrt{(ah)}\cot\theta+\frac{2}{3}a\cot^2\theta, 0, -2a\cot\theta]$. The join of these is perpendicular to the plane of flotation; therefore $h = \frac{1}{16}a(6\tan\theta+5\cot\theta)^2$.

Volume immersed $= 2a\pi \mathrm{PV}^2 = 2a\pi[h-2\sqrt{(ah)}\cot\theta+a\cot^2\theta]^2$; therefore

$$\text{specific gravity} = \left(\frac{h-2\sqrt{(ah)}\cot\theta+a\cot^2\theta}{h}\right)^2 = \left(\frac{1+5\sin^2\theta}{5+\sin^2\theta}\right)^4.$$

15682. (J. J. BARNIVILLE, B.A.)—Having $u_n+2u_{n+1}+u_{n+2} = u_{n+3}$, prove that

$$\frac{1.2.5}{1.2.3.4}+\frac{3.5.10}{1.4.6.9}+\frac{6.10.22}{3.9.13.19}+\frac{13.22.47}{6.19.28.41}+\ldots = \frac{3}{2};$$

$$\frac{2.3.7}{1.3.4.6}+\frac{4.7.15}{2.6.9.13}+\frac{9.15.32}{4.13.19.28}+\frac{19.32.69}{9.28.41.60}+\ldots = \frac{7}{6}.$$

Solution by C. M. ROSS.

1. The scale of relation of the series 2, 5, 10, 22, 47, ... is

$$u_n+2u_{n+1}+u_{n+2} = u_{n+3};$$

however, it is better to find the scale of relation of 1, 2, 3, 4, 6, 9, ... If $u_0, u_1, u_2, u_3, u_4, \ldots$ denote it, the required scale is $u_{n+3} = u_{n+2}+u_n$. Then

$$\frac{1.2.5}{1.2.3.4} = \frac{(u_1-u_0)(u_3-u_1)(u_3+u_0)}{u_0u_1u_2u_3}$$
$$= [u_1u_3(u_3-u_1)-u_0u_1(u_1-u_0)-u_0u_3(u_3+u_0)+2u_0u_1u_3]/u_0u_1u_2u_3$$
$$= [u_1u_3(u_2-u_0)-u_0u_1(u_3-u_2)-u_0u_3(u_1+u_2)+2u_0u_1u_3]/u_0u_1u_2u_3$$
$$= \frac{1}{u_0}-\frac{1}{u_1}-\frac{1}{u_2}+\frac{1}{u_3}.$$

Similarly $$\frac{3.5.10}{1.4.6.9} = \frac{1}{u_0}-\frac{1}{u_3}-\frac{1}{u_4}+\frac{1}{u_5},$$

and so on; therefore, by addition,

$$S_\infty = \frac{2}{u_0}-\frac{1}{u_1} = \frac{3}{2}.$$

2. Again $$\frac{2.3.7}{1.3.4.6} = \frac{1}{u_0}-\frac{1}{u_2}-\frac{1}{u_3}+\frac{1}{u_4},$$

$$\frac{4.7.15}{2.6.9.13} = \frac{1}{u_2}-\frac{1}{u_4}-\frac{1}{u_5}+\frac{1}{u_6},$$

and so on; therefore, by addition,

$$S_\infty = \frac{1}{u_0}+\frac{1}{u_4} = \frac{7}{6}.$$

15674. (W. F. BEARD, M.A.)—TP, TQ are tangents, and TAB a secant, to a circle; any circle through AB cuts BP, BQ at C, D. Prove that PQ bisects CD.

Solutions (I.) *by* R. F. DAVIS, M.A.; (II.) *by the* PROPOSER; (III.) *by* A. W. T.; (IV.) *by* M. V. A. SASTRY, B.A.

(I.) Let PQ, CD intersect in O. Since BCAD is cyclic, the angles ACD, ABD are equal. Thus the angles ACO, APO are also equal, and OAPC is cyclic. The triangles AOC, AQB are therefore directly similar, and OC : OA = QB : QA = TQ : TA. In precisely similar fashion it can be shown that OD : OA = TP : TA. Hence OC = OD.

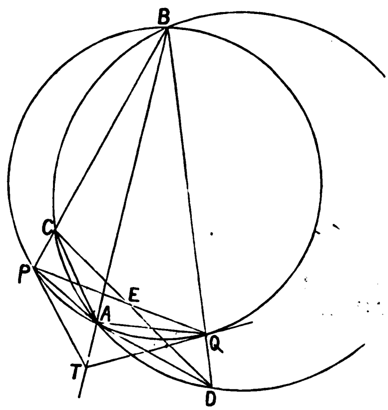
(II.) Join AC, AP, AQ, AD.

$\angle AQD = \angle APC$,

$\angle ADQ = \angle ACP$

(Euc. III. 22); therefore the triangles ADQ, ACP are similar; therefore

$DQ/AQ = CP/AP \ldots (1).$



From similar triangles ATP, PTB,

$AP/TP = PB/BT$,

and similarly

$AQ/TQ = QB/BT$.

But $TP = TQ$;

therefore

$AP/PB = AQ/QB$;

therefore, from (1),

$DQ/QB = CP/PB$.

But, by Menelaus' theorem, $CE/ED \cdot DQ/QB \cdot BP/PC = 1$; therefore $CE = ED$.

(III.) Let PQ, CD intersect in R. Then it is well known that the circles CPR, QDR intersect in A,

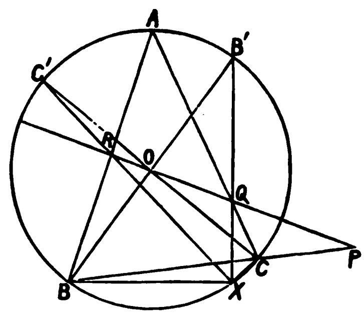
$$\frac{\Delta ACR}{\Delta ADR} = \frac{AC \sin CAR}{AD \sin DAR} = \frac{AC}{AD}\,\frac{\sin BPQ}{\sin BQR} = \frac{AP}{AQ}\,\frac{BQ}{BP} = 1;$$

therefore $CR = RD$.

(IV.) Let PA meet the circle ABC in E. Join DE, EC. Let PQ meet DC, EC in O, F respectively. Then we have to prove that O is the middle point of CD. Now $\angle TPA = \angle PBA = \angle AEC$; therefore TP is parallel to EC. Again, $\angle APQ = \angle ABQ = \angle AED$; therefore DE is parallel to PQ. (TAHB) is a harmonic range; therefore P (TAHB) is a harmonic pencil; therefore P (∞ EFC) is a harmonic pencil (because PT, CE meet at ∞); therefore (∞ EFC) is a harmonic range; therefore F is the mid-point of CE, and FO is parallel to ED; therefore O is the mid-point of CD; therefore PQ bisects CD.

15633. (JAMES BLAIKIE, M.A.)—If a straight line drawn through the circum-centre of a triangle ABC meet BC, CA, AB in P, Q, R, prove that the circles described on AP, BQ, CR as diameters concur in two points, one on the circum-circle, the other on the nine-point circle, and that their common chord passes through the orthocentre.

Solution by the PROPOSER.



Draw the diameters BOB′, COC′. Join B′Q, C′R and produce them to meet in X. Then, by Pascal's theorem (converse), since ABB′XC′C is a hexagon and since the intersections of the diagonals Q, O, R are collinear, X is a point on the circle ABC. The angles BXB′, CXC′ are right angles, being angles in semicircles. Therefore the semicircles on BQ, CR as diameters pass through X. Similarly the circle on AP as diameter passes through X. Let the circle BXQ meet AC in M and let CXR meet AB in N. Then BM, CN are perpendicular to AC, AB, and intersect in H. Let XH meet BXQ in Y. Then

$$HX.HY = BH.HM = CH.HN,$$

since B, M, N, C are concyclic; therefore Y is a point on CXR; that is, the circles meet on XH. Also XH.HY is constant; therefore Y is the inverse of X and its locus is a circle, since the locus of X is a circle. Also, when X coincides with A, B, C, Y coincides with the feet of the altitudes; therefore the locus of Y is the nine-point circle.

15310. (R. TUCKER, M.A.)—PF and QE are the radii of curvature at the extremities of a focal chord of a parabola. Show that PE and QF produced intersect on the hyperbola $2x^2 - y^2 = 4ax$.

Solution by the PROPOSER.

Let PF, QE be the radii of curvature; then the equations to QF, PE are $\quad y(3m^2+2-m'^2)+2(m^3+m')x = 4a(m'-m)$

and $\quad y(3m'^2+2-m^2)+2(m'^3+m)x = 4a(m-m')$,

with the relation $mm' = -1$. Adding, we get

$$y+(m'+m)x = 0.$$

Subtracting, $\quad 2y(m'+m)+x[(m'+m)^2+2] = 4a.$

Eliminating $m'+m$, we get $2x^2-y^2 = 4ax$.

14078. (Professor S. Sircom, M.A.)—If n is a positive integer, prove that

$$\int_0^\infty \frac{\sin^n\theta}{\theta^n}\,d\theta = \frac{1}{(n-1)!\,2^n}\left\{n^{n-1}-n(n-2)^{n-1}+\frac{n(n-1)}{1.2}(n-4)^{n-1}\right.$$
$$\left.-\frac{n(n-1)(n-2)}{1.2.3}(n-6)^{n-1}+\dots\right\}\pi.$$

[I have only seen this theorem worked for low values of n.]

Additional Solution and Remarks by Professor Nanson.

Integrating by parts, we have

$$\int_0^\infty \frac{\sin^n\theta}{\theta^n}\,d\theta = \frac{1}{(n-1)!}\int_0 \frac{1}{\theta}\left(\frac{d}{d\theta}\right)^{n-1}\sin^n\theta\,d\theta,$$

the integrated terms clearly vanishing at the limits. Now

$$\sin^n\theta = \frac{1}{2^{n-1}}\,\Sigma n_r\cos(n-2r)(\theta-\tfrac{1}{2}\pi)+\text{const.};$$

therefore $$\left(\frac{d}{d\theta}\right)^{n-1}\sin^n\theta = \frac{1}{2^{n-1}}\,\Sigma(-1)^r n_r(n-2r)^{n-1}\sin(n-2r)\,\theta;$$

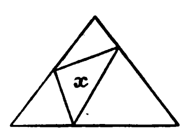
therefore $$\int_0^\infty \frac{\sin^n\theta}{\theta^n}\,d\theta = \frac{\pi}{(n-1)!\,2^n}\,\Sigma(-1)^r n_r(n-2r)^{n-1}.$$

Another solution is given in *Reprint*, Vol. LXXI., pp. 115–7, in regard to which reference should be made to a paper by Dr. Glaisher, *Proc. London Math. Soc.*, Vol. IV., pp. 291–302.

15687. (R. Chartres.)—Three random points are taken in the sides of a triangle, one in each side, and joined. Find the mean value of the square of the area of the triangle thus formed. Elementary proof wanted.

Solution by the Proposer.

Let
$$\Delta - x = y,$$
$$M(x^2) = Mx(\Delta-y)$$
$$= \tfrac{1}{4}\Delta^2-\tfrac{1}{6}\Delta^2$$
$$= \tfrac{1}{12}\Delta^2.$$



15683. (Professor Langhorne Orchard, M.A., B.Sc.)—Find the product of n terms of the series $2+34+246+1028+3130+\dots$ by n terms of the series $0+30+240+1020+3120+\dots$?

Solution by C. M. Ross *and* Kela Deva Rau, B.A.

The first series may be written

$$1 + 1^5 + (2 + 2^5) + (3 + 3^5) + (4 + 4^5) + (5 + 5^5) + \dots = \Sigma n^5 + \Sigma n.$$

Similarly, the second series is equal to

$$1^5 - 1 + (2^5 - 2) + (3^5 - 3) + (4^5 - 4) + (5^5 - 5) + \dots = \Sigma n^5 - \Sigma n.$$

If S and S_1 are the required sums of above,

$$SS_1 = (\Sigma n^5)^2 - (\Sigma n)^2 = [\tfrac{1}{12} n^2 (n+1)^2 (2n^2 + 2n - 1)]^2 - [\tfrac{1}{2} n (n+1)]^2$$
$$= \tfrac{1}{144} n^2 (n+1)^2 [n^2 (n+1)^2 (2n^2 + 2n - 1)^2 - 36],$$

which is the required product.

15564. (A. M. Nesbitt, M.A.)—The normals at the extremities of a chord of an ellipse meet in the curve. Prove that the pole of the chord lies on a concentric ellipse.

Solution by Frances E. Cave *and* W. F. Beard, M.A.

(I.) Let (h, k) be the intersection of the normals, (ξ, η) the pole. Then the polar of (ξ, η) and a certain chord through (h, k) form a conic through the four feet of the normals; therefore

$$x^2/a^2 + y^2/b^2 - 1 + \lambda[(a^2 - b^2) xy - a^2 hy + b^2 kx]$$
$$\equiv (x\xi/a^2 + y\eta/b^2 - 1)[A(x - h) + B(y - k)]$$

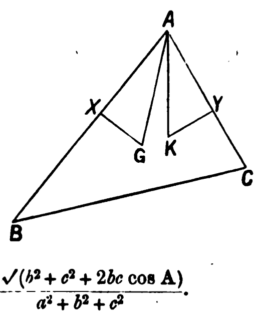
for suitable values of λ, A, B. Equating coefficients and eliminating,

$$(a^2 - b^2)/(a^2\eta^2 + b^2\xi^2) = k/[\eta(\xi^2 - a^2)] = -1/(\xi^2 - \eta^2 - a^2 + b^2);$$

therefore the pole lies on $a^2x^2 + b^2y^2 = (a^2 - b^2)^2$.

10041. (Professor Emmerich, Ph.D.)—K being the symmedian point of the triangle ABC, we have $AK + BK + CK \equiv (a + b + c)/\sqrt{3}$.

Remarks by Professor Sanjána, M.A., *and others.*



The result is inaccurate. Drawing the perpendiculars from G and K to AB and AC respectively, we have the triangles AGX, AKY similar, as G and K are isogonal conjugates. Hence

$$AK/AG = KY/GX = \lambda b/(\mu/C);$$

therefore

$$AK = (\lambda/\mu)\, bc \,.\, AG = \tfrac{2}{3}(\lambda/\mu)\, bcm_1$$

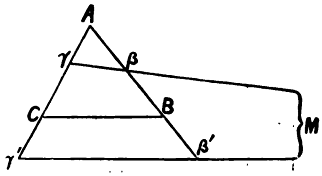
where m_1 is the median from A. Also $\lambda = \frac{1}{2}\tan\omega$, $\mu = \frac{2}{3}\Delta$; so that, finally,

$$AK = \frac{1}{2}\,\frac{\tan\omega}{\Delta}\, bcm_1 = \frac{2bcm_1}{a^2 + b^2 + c^2} = \frac{bc\sqrt{(b^2 + c^2 + 2bc\cos A)}}{a^2 + b^2 + c^2}.$$

Hence $\quad \Sigma AK = [\Sigma bc\sqrt{(b^2 + c^2 + 2bc\cos A)}]/(a^2 + b^2 + c^2)$,

which is not equal to $\Sigma a/\sqrt{3}$ when the triangle is general.

15685. (Professor COCHEZ.)—On donne un triangle ABC; on porte sur AB et AC les segments $A\beta$, $A\gamma$ tels que $A\beta/A\gamma = K$, puis $B\beta' = A\beta$ et $C\gamma' = A\gamma$. Trouver le lieu du point M de rencontre des droites $\beta\gamma$ et $\beta'\gamma'$.



Solution by R. TUCKER, M.A., *and* A. M. NESBITT, M.A.

Take AB, AC for axes and put $A\beta = h$, $A\gamma = k$; then

$$x/h + y/k = 1, \quad x/(c+h) + y/(b+k) = 1, \quad h = k\lambda \ldots \text{(i., ii., iii.)}.$$

From (i.) and (ii.), $cx/(c+h) + b\lambda y/(b+k) = 0$; and, from (i.), $x + \lambda y = k\lambda$; whence, eliminating k, we get $(x+\lambda y)(bc\lambda + cx + b\lambda^2 y) = 0$; whence the locus is $x/b\lambda + \lambda y/c = -1$.

15610. (C. M. ROSS.)—Eliminate x, y, z from the equations

$$x^2+y^2+z^2 = l^2, \quad x^3+y^3+z^3 = m^3, \quad x^4+y^4+z^4 = n^4, \quad xyz = p^3 \quad (1, 2, 3, 4).$$

Solutions (I.) *by* R. F. DAVIS, M.A., *and* A. M. NESBITT, M.A.; (II.) *by the* PROPOSER.

(I.) Assume $x+y+z = t$. Then $m^3 - 3p^3 = \frac{1}{2}t\,(3l^2 - t^2)$, and

$$t^3 - 3l^2t + 2\,(m^3 - 3p^3) = 0 \ldots\ldots\ldots\ldots\text{(i.)}.$$

Again $2\Sigma yz = t^2 - l^2$, $4\,(\Sigma y^2z^2 + 2p^3t) = t^4 - 2l^2t^2 + l^4$. But $2\Sigma y^2z^2 = l^4 - n^4$; wherefore

$$t^4 - 2l^2t^2 - 8p^3t - l^4 + 2n^4 = 0 \ldots\ldots\ldots\ldots\text{(ii.)}.$$

From (i.) and (ii.) we can eliminate t, the result being apparently of the 12th degree in l, m, n, p.

(II.) Let x, y, z be the roots of the cubic $\lambda^3 - a\lambda^2 + b\lambda - c = 0$; then

$$\Sigma(x) = a, \quad \Sigma(yz) = b, \quad xyz = c;$$

also $\Sigma(x^2) = a^2 - 2b$, $\Sigma(x^3) = a^3 - 3ab + 3c$, $\Sigma(x^4) = a^4 - 4a^2b + 4ac + 2b^2$.

Hence the problem reduces to eliminating a, b, c from the equations

$$a^2 - 2ab = l^2, \quad a^3 - 3ab + 3c = m^3, \quad a^4 - 4a^2b + 4ac + 2b^2 = n^4 \ldots (5, 6, 7),$$

$$c = p^3 \ldots\ldots\ldots\ldots (8).$$

From (5), $b = \frac{1}{2}(a^2 - l^2)$; (6) then becomes

$$a^3 - 3al^2 + 2m^3 - 6p^3 = 0 \ldots\ldots\ldots\ldots (\alpha),$$

and (7) likewise $\quad a^4 - 2a^2l^2 - 8ap^3 + 2n^4 - l^4 = 0 \ldots\ldots\ldots\ldots (\beta).$

Now a can be eliminated from (α) and (β) by means of quadratics or by Sylvester's method. The last is the best, as rationalizing quantities is avoided. Multiplying (α) by a, a^2, a^3, and (β) by a and a^2 in succession, the result is

$$\begin{aligned}
a^3 - 3al^2 + 2m^3 - 6p^3 &= 0,\\
a^4 - 3a^2l^2 + 2\,(m^3 - 3p^3)\,a &= 0,\\
a^5 - 3a^3l^2 + 2\,(m^3 - 3p^3)\,a^2 &= 0,\\
a^6 - 3a^4l^2 + 2\,(m^3 - 6p^3)\,a^3 &= 0,\\
a^4 - 2a^2l^2 - 8ap^3 + 2n^4 - l^4 &= 0,\\
a^5 - 2a^3l^2 - 8a^2p^3 + (2n^4 - l^4)\,a &= 0,\\
a^6 - 2a^4l^2 - 8a^3p^3 + (2n^4 - l^4)\,a^2 &= 0.
\end{aligned}$$

Eliminating $a^6, \ldots, a$ from these equations,

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 0 & -3l^2 & 2(m^3-3p^3) \\ 0 & 0 & 1 & 0 & -3l^2 & 2(m^3-3p^3) & 0 \\ 0 & 1 & 0 & -3l^2 & 2(m^3-3p^3) & 0 & 0 \\ 1 & 0 & -3l^2 & 2(m^3-3p^3) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2l^2 & -8p^3 & 2n^4-l^4 \\ 0 & 1 & 0 & -2l^2 & -8p^3 & 2n^4-l^4 & 0 \\ 1 & 0 & -2l^2 & -8p^3 & 2n^4-l^4 & 0 & 0 \end{vmatrix} = 0,$$

which reduces to

$$\begin{vmatrix} 1 & 0 & -3l^2 & 2m^3-3p^3 \\ 0 & -l^2 & 2(m^3+p^3) & l^4-2n^4 \\ -l^2 & 2(m^3+p^3) & l^4-2n^4 & 0 \\ 2(m^3+p^3) & -(l^4+2n^4) & -8l^2p^3 & l^2(2n^4-l^4) \end{vmatrix} = 0,$$

the required eliminant.

15667. (Professor NEUBERG.) — Chercher la condition pour que les équations $\tan x = a \tan(y-z)$, $\tan y = b \tan(z-x)$, $\tan z = c \tan(x-y)$ soient compatibles.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Write (for shortness) $\tan x = \xi$, $\tan y = \eta$, $\tan z = \zeta$, $a^{-1} = \alpha$, $b^{-1} = \beta$, $c^{-1} = \gamma$, $\alpha+\beta+\gamma = -\mu$. Then, expressing the tangents of $y-z$, $z-x$, $x-y$ in terms of ξ, η, ζ, and reducing fractions,

$$(1+\eta\zeta)\,\xi = a(\eta-\zeta), \quad (1+\zeta\xi)\,\eta = b(\zeta-\xi), \quad (1+\xi\eta)\,\zeta = c(\xi-\eta).$$

Hence $\quad \xi\eta\zeta = -\xi + a\eta - a\zeta = -b\xi - \eta + b\zeta = c\xi - c\eta - \zeta.$

Hence also $\quad \mu\,.\,\xi\eta\zeta = \alpha\xi + \beta\eta + \gamma\zeta.$

Eliminating (ξ, η, ζ) gives three linear equations homogeneous in ξ, η, ζ; so that ξ, η, ζ may be eliminated in the form below (which is the required result)

$$\begin{vmatrix} \alpha+\mu, & \beta-\mu a, & \gamma+\mu a \\ \alpha+\mu b, & \beta+\mu, & \gamma-\mu b \\ \alpha-\mu c, & \beta+\mu c, & \gamma+\mu \end{vmatrix} = 0.$$

15598. (R. TUCKER, M.A.)—PSQ is a fixed focal chord of a parabola, and R a variable point on the curve. Find (i.) the locus of the orthocentre (K) of the triangle PRQ, and (ii.) the envelope of the locus of K, supposing the chord to vary in position.

Solution (I.) *by* FRANCES E. CAVE *and* W. F. BEARD, M.A.; (II.) *by* Professor SANJÁNA, M.A.

(I.) Let $y = m(x-a)$ be the equation of PQ; y_1, y_2, y_3 ordinates of P, Q, R; then $y_1 y_2 = -4a^2$ and $y_1 + y_2 = 4a/m$. K is the intersection of

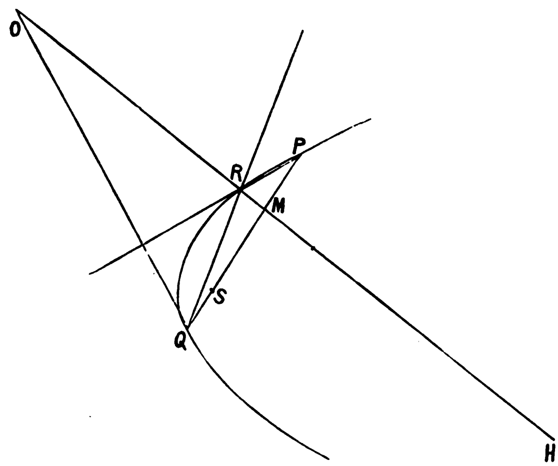
$4a(y-y_2)+(y_1+y_3)(x-y_3^2/4a)=0$ and a similar line. Eliminating y_3, $(x-a)(x+3a)-4ay/m-4ax/m^2=0$ is the equation of the locus of K. The equation of the envelope when m varies is $x(x-a)(x+3a)+ay^2=0$.

Mr. Beard also solves the first part of the problem geometrically as follows:—

Let O be the orthocentre of the triangle PQR. Let OR meet PQ at M and the parabola again at H. Let K be the focal chord perpendicular to PQ. Then PM . MQ : RM . MH = PQ : K. But

$$PM . MQ = OM . MR;$$

therefore OM . MR : MR . MH = PQ : K; therefore the ratio OM : MH is constant and H is on the parabola; therefore the locus of O is also a para-



bola. If PQ < K, then, if through PQ we draw a plane at an angle $\cos^{-1}$ PQ/K with the given plane, the locus of O is equal to the projection of the given parabola on this plane. If PQ > K, and we draw a plane through QP at an angle $\cos^{-1}$ K/PQ with the given plane, then the projection of a curve equal to the locus of O is the given parabola.

(II.) Let P be the point $(a\cot^2\theta, 2a\cot\theta)$; then Q is $(a\tan^2\theta, -2a\tan\theta)$; suppose R to be $(a\cot^2\phi, 2a\cot\phi)$. The equation of PR is

$$y(\cot\theta+\cot\phi)=2x+2a\cot\theta\cot\phi;$$

and therefore that of QK, the perpendicular on PR, is

$$y+2a\tan\theta=-\tfrac{1}{2}(\cot\theta+\cot\phi)(x-a\tan^2\theta).$$

Similarly, the equation of QR is

$$y(-\tan\theta+\cot\phi)=2x-2a\tan\theta\cot\phi;$$

and of the perpendicular on it from P,

$$y-2a\cot\theta=\tfrac{1}{2}(\tan\theta-\cot\phi)(x-a\cot^2\theta).$$

Hence, eliminating ϕ, we have at the orthocentre K

$$(y-2a\cot\theta)/(x-a\cot^2\theta)-(y+2a\tan\theta)/(x-a\tan^2\theta)=\tfrac{1}{2}(\cot\theta+\tan\theta),$$

i.e., $$x^2-ax(\cot^2\theta+\tan^2\theta-4)-2ay(\cot\theta-\tan\theta)-3a^2=0;$$

so that K moves on a parabola.

Putting $\cot\theta - \tan\theta = \lambda$, we may write the above thus

$$x^2 - ax(\lambda^2-2) - 2ay\lambda - 3a^2 = 0 \quad \text{or} \quad ax\lambda^2 + 2ay\lambda - (x^2+2ax-3a^2) = 0\,;$$

hence when θ varies the required envelope is

$$4a^2y^2 + 4ax(x^2+2ax-3a^2) = 0 \quad \text{or} \quad ay^2 + x(x-a)(x+3a) = 0,$$

which is a cubic through the vertex.

15655. (Professor Cochez.)—Trouver le lieu des points qui divisent en moyenne et extrême raison les cordes d'une ellipse passant par un point fixe.

Solution by Professor Sanjána, M.A.

Take the fixed point O for origin, and let the curve be

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If any chord be drawn at an angle θ with the x-axis, the distances r_1, r_2 of its extremities P, Q from the origin are the roots of the equation

$$r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) + 2r(g\cos\theta + f\sin\theta) + c = 0.$$

Take R to divide PQ in extreme and mean ratio, so that PQ . RQ = PR^2, *i.e.*, $(\mathrm{OQ}-\mathrm{OP})(\mathrm{OQ}-\rho) = (\rho-\mathrm{OP})^2$, where ρ stands for OR. Hence

$$\rho^2 - 3\rho r_1 + \rho r_2 - r_2^2 + r_1^2 + r_1 r_2 = 0.$$

If we put $\mathrm{D} \equiv a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta$, we get

$$r_1 = -1/\mathrm{D}\,\{g\cos\theta + f\sin\theta + \surd[(g\cos\theta+f\sin\theta)^2 - c\mathrm{D}]\}$$

and $$r_2 = -1/\mathrm{D}\,\{g\cos\theta + f\sin\theta - \surd[(g\cos\theta+f\sin\theta)^2 - c\mathrm{D}]\}.$$

Substituting, we obtain

$$\rho^2 + 2\rho(g\cos\theta+f\sin\theta)/\mathrm{D} + c/\mathrm{D} + 4\rho\surd[(g\cos\theta+f\sin\theta)^2 - c\mathrm{D}]/\mathrm{D}$$
$$+\,4(g\cos\theta+f\sin\theta)\surd[(g\cos\theta+f\sin\theta)^2 - c\mathrm{D}]/\mathrm{D}^2 = 0.$$

Multiply out by D, and for $\rho\cos\theta$, $\rho\sin\theta$ put x, y respectively; hence the locus is seen to be given by the equation

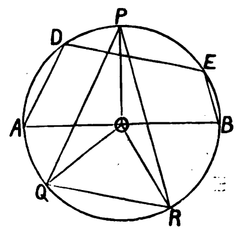
$$\mathrm{S} + 4\surd[(gx+fy)^2 - c(ax^2+2hxy+by^2)]$$
$$+\frac{4(gx+fy)\surd[(gx+fy)^2 - c(ax^2+2hxy+by^2)]}{ax^2+2hxy+by^2} = 0.$$

On rationalizing this equation is seen to be of the 8th degree.

15622. (Professor Sanjána, M.A.)—In a circle, whose diameter is AB, a quadrilateral ADEB is inscribed, and PQR is the inscribed triangle whose sides are parallel to AD, DE, EB, the side QR being parallel to DE. Prove that QR^2+DE^2 = AB^2, and that the triangle is equal in area to the quadrilateral.

Solutions (I.) *by* R. F. DAVIS, M.A., *and* C. SEARLE; (II.) *by* W. F. BEARD, M.A., *and others;* (III.) *by* R. TUCKER, M.A.

(I.) From any point P draw PQ, PR parallel to DA, EB respectively, so that the arcs AQ, DP, and also BR, EP, are equal. Then QR will be parallel to DE if the arcs DQ, ER are equal; that is, if the arcs AP, BP are equal.

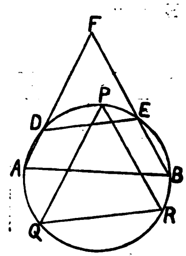


Thus OP is perpendicular to AOB; also OQ to OD and OR to OE. Since DE, QR subtend supplementary angles at O,

$$DE^2 + QR^2 = AB^2.$$

Moreover, the quadrilateral ADEB = sum of the triangles AOD, DOE, EOB = sum of the triangles POQ, QOR, ROP = ΔPOQ. For the angles AOD, DOB are supplementary; and the arcs DP, PB are together equal to the arcs AQ, AP; &c.

(II.) Produce AD, BE to meet at F. The arc DQ is equal to the arc ER, since DE is parallel to QR, and is equal to the arc PB, since EB is parallel to PR; therefore the arc PDQ = the arc DPB; therefore PQ = DB. The triangles PQR, FDE, FBA are equiangular; therefore



$$\Delta PQR : \Delta FDE : \Delta FBA$$
$$= PQ^2 : FD^2 : FB^2 = DB^2 : FD^2 : FB^2.$$

But $\qquad FB^2 = DB^2 + FD^2 \qquad$ (Euc. I. 47); therefore

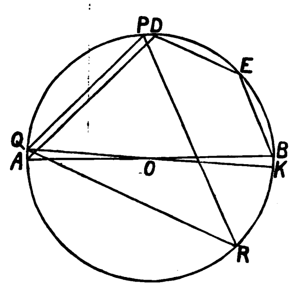
$$\Delta PQR = \Delta FBA - \Delta FDE = \text{quadl. ADEB}\ldots(1).$$

Also

$$\Delta PQR : \Delta FDE : \Delta FBA = QR^2 : DE^2 : AB^2;$$

therefore, from (1), $\qquad QR^2 + DE^2 = AB^2.$

(III.) Draw the diameter QOK. Then BK = AQ = PD; therefore DE = RK. Hence



$$QR^2 + DE^2 = QR^2 + RK^2 = \ldots.$$

Again,

$$\text{arc PK} = \text{arc PB} + \text{arc BK}$$
$$= \text{arc RE} + \text{arc BK}$$
$$= \text{arc DQ} + \text{arc AQ} = \text{arc AD}.$$

Hence $PK^2 + PQ^2 = PQ^2 + AD^2 = \ldots.$

Drop perpendiculars p, p' on QR, DE; then $QR^2 = 4(R^2 - p^2)$, and therefore $2p = DE$. $DE^2 = 4(R^2 - p'^2)$, and therefore $2p' = QR$.

Now $\qquad 2\Delta PQR = p\,.\,QR + \ldots = \tfrac{1}{2}QR\,.\,DE + \ldots$

and $\qquad 2\text{ quad.} = p'\,.\,DE + \ldots = \tfrac{1}{2}QR\,.\,DE + \ldots,$

14987. (T. MUIR, M.A., F.R.S.)—Given

$$u \equiv (a,\ b,\ c,\ d \between x,\ y)^3 + e = 0,$$

show that
$$\frac{d^2y}{dx^2}\left(\frac{du}{dx}\right)^3 = 2e\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ y^2, & -xy, & x^2 \end{vmatrix},$$
and generalise.

Extended Solution by Professor NANSON.

A generalization follows from the known theorem that, if y_r be determined as a function of $y_1, y_2, \ldots, y_{r-1}, y_{r+1}, \ldots, y_m$ by the equation $\phi(y_1, y_2, \ldots, y_m) = 0$, and H_r is the Hessian of y_r, whilst K is the Hessian of ϕ bordered with the first differential coefficients of ϕ, then

$$(d\phi/dy_r)^{m+1}\,H_r = (-1)^m\,K.$$

If, now, ϕ is of the form $v+e$ where v is an m-ary n-ic and e is a constant, we have, by a well known theorem, $K = -nv/(n-1)\,H$ where H is the Hessian of v. Thus, if y_r be determined as function of the other variables by the equation $v+e=0$ where v is an m-ary n-ic, the Hessian H_r of y_r is determined by

$$(dv/dy_r)^{m+1}\,H_r = (-1)^m\,ne/(n-1)\,H$$

where H is the Hessian of v.

When v is a binary quantic it is known that $(n-2)^2\,H = \Delta$ where Δ denotes the determinant

$$\begin{vmatrix} d^3v/dx^3 & d^3v/dx^2dy & d^3v/dx\,dy^2 \\ d^3v/dx^2dy & d^3v/dx\,dy^2 & d^3v/dy^3 \\ y^2 & -xy & x^2 \end{vmatrix},$$

and thus we have $(dv/dy)^3\,d^2y/dx^2 = ne\Delta/[(n-1)(n-2)^2]$, which is the extension of Dr. Muir's result to any binary quantic.

To obtain the extension to any quantic it may be observed that, if J be the Jacobian of the three ternary quadrics

$$(a_r\,b_r\,c_r\,f_r\,g_r\,h_r)(xyz) \quad (r = 1, 2, 3),$$

then the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & f_1 & g_1 & h_1 \\ a_2 & b_2 & c_2 & f_2 & g_2 & h_2 \\ a_3 & b_3 & c_3 & f_3 & g_3 & h_3 \\ . & z^2 & y^2 & -yz & . & . \\ z^2 & . & x^2 & . & -zx & . \\ y^2 & x^2 & . & . & . & -xy \end{vmatrix}$$

has the value $-2^{-3}xyzJ$, and the analogous determinant formed for m m-ary quadrics has the value $(-1)^{\frac{1}{2}m(m-1)}\,2^{-m}\,(xyz\ldots)^{m-2}\,J$ where J is the Jacobian of the quadrics. Now, taking the quadrics to be the differential coefficients of a cubic, the Jacobian J becomes the Hessian of the cubic, and we at once obtain the extension of Dr. Muir's result for any cubic. But, further, if in the Hessian thus calculated we replace the coefficients of the cubic by the third differential coefficients of a quantic and divide the determinant by $(n-2)^m$, we readily obtain the extension of Dr. Muir's result to any quantic.

15582. (W. SCRIMGEOUR, M.A., B.Sc.)—QSQ′ is a focal chord of a conic. PG, the normal at a point P on the curve, is perpendicular to QSQ′, and meets the axis in G. Prove that $QS \cdot Q'S = PG^2$.

Solutions (I.) *by* F. W. REEVES, B.A., *and others*; (II.) *by* C. M. ROSS *and* C. A. B. GARRETT; (III.) *by* R. F. DAVIS, M.A.; (IV.) *by* R. TUCKER, M.A.

(I.) Fig. 1 (*Parabola*).—The figures explain themselves. Because QSQ is the focal chord, and QT, Q′T are tangents, the angles QTQ′, QST are right angles. Also PG is perpendicular to the tangent at P and also to QSQ′, *i.e.*, parallel to ST, and PT is parallel to SG; therefore

$$PG^2 = ST^2 = VT^2 - SV^2 = QV^2 - SV^2 = QS \cdot SQ'.$$

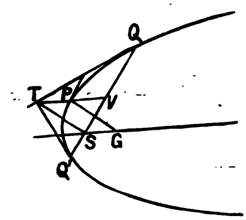


FIG. 1.

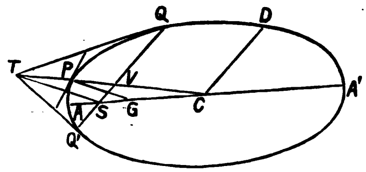
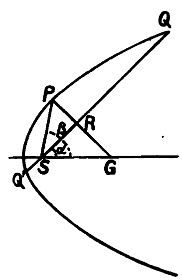


FIG. 2.

Fig. 2 (*Central Conic*).—Because QQ′, AA′ intersect in S,

$$QS \cdot Q'S : AS \cdot A'S = CD^2 : CA^2 = PG^2 : CB^2.$$

But $AS \cdot A'S = CB^2$; therefore $QS \cdot Q'S = PG^2$.



(II.) Using polar co-ordinates and denoting the points Q, Q′, P by (r_1, a_1), $(r_2, 180° + a)$, (r_3, β),

$$r_1 = l/(1-\epsilon\cos a), \quad r_2 = l/(1+\epsilon\cos a),$$

$$r_3 = l/(1-\epsilon\cos\beta);$$

therefore $QS \cdot Q'S = r_1 r_2 = l^2/(1-\epsilon^2\cos^2 a)$.

The equation of the normal at P in Cartesians is

$$\frac{1-\epsilon\cos\beta}{\epsilon}x + \frac{(1-\epsilon\cos\beta)(\epsilon-\cos\beta)}{\epsilon\sin\beta} = \frac{l}{r} \ldots (1).$$

The equation of the focal chord is

$$y = x\tan a \ldots\ldots\ldots\ldots (2).$$

(1) and (2) are perpendicular by hypothesis; therefore $\tan a \sin\beta = \cos\beta - \epsilon$; therefore

$$\cos^2\beta - 2\epsilon\cos^2 a\cos\beta + \epsilon^2\cos^2 a - \sin^2 a = 0;$$

therefore $$\cos\beta = \epsilon\cos^2 a \pm \sin a\,(1-\epsilon^2\cos^2 a)^{\frac{1}{2}}.$$

Now $$SG = SR\sec a = r_3\cos(\beta - a)\sec a;$$

therefore $PG^2 = SG^2 + SP^2 - 2SG \cdot SP\cos\beta = l^2\sin^2\beta/\cos^2 a\,(1-\epsilon\cos\beta)^2$ (on simplification). Again

$$\sin^2\beta = \cos^2 a\left[(1-\epsilon^2\cos^2 a)^{\frac{1}{2}} - \epsilon\sin a\right]^2$$

and $$(1-\epsilon\cos\beta)^2 = (1-\epsilon^2\cos^2 a)\left[(1-\epsilon^2\cos^2 a)^{\frac{1}{2}} - \epsilon\sin a\right]^2;$$

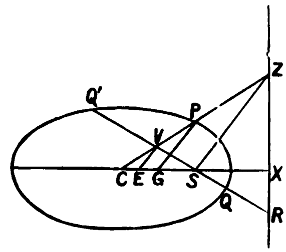
therefore $$PG^2 = l^2/(1-\epsilon^2\cos^2 a) = QS \cdot Q'S.$$

(III.) Draw SZ at right angles to QSQ′ to meet the S directrix in Z; then CZ passes through P and bisects QQ′ in V, say. Draw VE parallel to PG. Since

$$CP^2 = CV.CZ,$$

therefore

$$PG^2 = VE.ZS = VS.SR$$
$$\text{(for } VE : VS = SR : ZS)$$
$$= VS.VR - VS^2 = VQ^2 - VS^2$$
$$= QS.Q'S.$$



(IV.) Let $\angle QSG = \theta$; then $SQ.SQ' = l^2/(1-e^2\cos^2\theta)$...............(i.).
Now $\tan\theta = b\cot\phi/a$, where ϕ is the eccentric angle of P,

$$\sec^2\theta = (a^2\sin^2\phi + b^2\cos^2\phi)/(a^2\sin^2\phi),$$

and $$PG^2 = b^2(b^2\cos^2\phi + a^2\sin^2\phi)/a^2 = b^2(1-e^2\cos^2\phi)\text{.........(ii.).}$$

Now (i.) is $l^2/\left\{1-[e^2a^2\sin^2\phi/(a^2\sin^2\phi+b^2\cos^2\phi)]\right\}$

$$= l^2a^2(1-e^2\cos^2\phi)/b^2 = b^2(1-e^2\cos^2\phi) = PG^2.$$

15481. (Lt.-Col. Allan Cunningham, R.E.)—Factorize completely $N = 96^{18}+1$.

Additional Solution by Professor Sanjána, M.A.

I have not seen Lucas's work referred to by the Proposer, but the following is a simple method of getting the result:—

Let $6x$ be of the form y^2; then

$$x^6+1 = (x^2+1)(x^4-x^2+1) = (x^2+1)\left\{(x^2+3x+1)^2-6x(x+1)^2\right\}$$
$$= (x^2+1)\left\{x^2+3x+1-y(x+1)\right\}\left\{x^2+3x+1+y(x+1)\right\},$$

and

$$x^{18}+1 = (x^6+1)(x^{12}-x^6+1) = \text{(product of factors above)}$$
$$\times\left\{x^6+3x^3+1-xy(x^3+1)\right\}\left\{x^6+3x^3+1+xy(x+1)\right\}.$$

In the present case, $x = 96$ and $y = 24$; so that we have the following factors of N:—

$$96^2+1 = 9217;\quad 96^2+3.96+1-24.97 = 7177;$$
$$96^2+3.96+1+24.97 = 11833;$$
$$96^6+3.96^3+1-2304(96^3+1) = 780{,}722{,}009{,}857;$$
$$96^6+3.96^3+1+2304(96^3+1) = 784{,}798{,}877{,}953.$$

Now $9217 = 13.709$, and 7177, 11833 are primes. It therefore remains to test the last two numbers, and this the Proposer has done (*Reprint*, New Series, Vol. VI., p. 63). We thus finally obtain

$$96^{18}+1 = 13.709.7177.11833.37.397.53150113.73.613.17537797.$$

I cannot offer any opinion on the primeness of the two large factors.

N.B.—The Proposer remarks that this factorisation is practically the same as Lucas's.

Note.—For the Proposer's Solution see *Reprint*, New Series, Vol. VI., p. 62.

15708. (*Communicated by* A. V. Kutti Krishna Menon, B.A.)—Prove that

$$\cos ax = 1 - ax\sin bx - [a(a-2b)/2!]x^2\cos 2bx + [a(a-3b)^2/3!]x^3\sin 3bx + [a(a-4b)^3/4!]x^4\cos 4bx - \dots.$$

[*Note.*—The Proposer desires to obtain an elegant solution.]

Solution by K. S. Patrachan.

Burmann's theorem gives us for the expansion of any function $f(z)$ in terms of any other function $F(z)$

$$f(z) = f(c) + \sum_{r=1}^{r=\infty} \frac{B_r\{F(z)\}^r}{r!},$$

where
$$B_r = \left[\frac{d^{r-1}}{dz^{r-1}}\left\{\left(\frac{z-c}{F(z)}\right)^r f'(z)\right\}\right]_{z=c}$$

and c is a root of the equation $F(z) = 0$. (See Edwards's *Differential Calculus.*) Putting $f(z) = e^{az}$ and $F(z) = ze^{bz}$, we have $c = 0$, and

$$\{(z-c)/F(z)\}^r f'(z) = ae^{(a-br)z};$$

therefore

$$B_r = \{d^{r-1}/dz^{r-1}[ae^{(a-br)z}]\}_{z=0} = [a(a-br)^{r-1}e^{(a-br)z}]_{z=0} = a(a-br)^{r-1}$$

and
$$f(c) = e^0 = 1.$$

Therefore

$$e^{az} = 1 + \sum_{r=1}^{r=\infty} \frac{a(a-br)^{r-1}(ze^{bz})^r}{r!}$$

$$= 1 + \frac{a}{1!}ze^{bz} + \frac{a(a-2b)}{2!}(ze^{bz})^2 + \frac{a(a-3b)^2}{3!}(ze^{bz})^3 + \frac{a(a-4b)^3}{4!}(ze^{bz})^4 + \dots.$$

In this result substitute successively for z, ιx and $-\iota x$ ($\iota = \sqrt{-1}$); we have

$$e^{\iota ax} = 1 + \frac{a}{1!}\iota xe^{\iota bx} + \frac{a(a-2b)}{2!}\iota^2x^2e^{2\iota bx} + \frac{a(a-3b)^2}{3!}\iota^3x^3e^{3\iota bx} + \frac{a(a-4b)^3}{4!}\iota^4x^4e^{4\iota bx} + \dots$$

and

$$e^{-\iota ax} = 1 + \frac{a}{1!}(-\iota x)e^{-\iota bx} + \frac{a(a-2b)}{2!}(-\iota x)^2e^{-2\iota bx} + \frac{a(a-3b)^2}{3!}(-\iota x)^3e^{-3\iota bx} + \frac{a(a-4b)^3}{4!}(-\iota x)^4e^{-4\iota bx} + \dots.$$

Therefore, adding these two results, we have

$$2\cos ax = 2 - 2\frac{a}{1!}x\sin bx - 2\frac{a(a-2b)}{2!}x^2\cos 2bx + 2\frac{a(a-3b)^2}{3!}x^3\sin 3bx + \dots,$$

since $e^{\iota\theta} + e^{-\iota\theta} = 2\cos\theta$ and $e^{\iota\theta} - e^{-\iota\theta} = 2\iota\sin\theta$, and on division by 2, we get the required result.

14907. (Professor A. Droz-Farny.)—Soit Σ une conique inscrite dans un triangle ABC. La tangente à Σ parallèle au côté BC et la tangente issue du milieu de BC se coupent en α. On obtient de même deux points analogues β et γ. Démontrer que les trois points α, β et γ sont en ligne droite.

Solution by Professor NANSON.

Consider the more general problem in which the tangents from a, ... cut BC, ... in points X, X′, ..., such that X, Y, Z are collinear and X, X′ are harmonic conjugates to B, C; ....

Expressing that the intersections X, X′ of the tangents

$$l/(yz'-y'z)+m/(zx'-z'x)+n/(xy'-x'y)=0$$

from xyz to an in-conic are harmonic to B, C, we have

$$-lx+my+nz=0 \quad\quad (1),$$

and expressing that X or X′ is on $\lambda x'+\mu y'+\nu z'=0$, we have

$$\mu^2 ny+\nu^2 mz=0 \quad\quad (2).$$

(1) and (2) determine a, and hence a, β, γ lie on the line

$$\frac{l^2}{\lambda^2}\left\{-lx+my+nz\right\}+\frac{m^2}{\mu^2}\left\{lx-my+nz\right\}+\frac{n^2}{\nu^2}\left\{lx+my-nz\right\}=0.$$

Note.—For another solution, see *Reprint*, New Series, Vol. I., p. 75.

7879. (D. EDWARDES.)—In any spherical triangle, prove that

$$2\sin s\sec^2 r=\sin c\cos(s-c)+\sin b\cos(s-b)+\sin a\cos(s-a),$$

r being the inscribed radius and $2s$ the perimeter.

Solution by Professor SANJÁNA, M.A., J. WEIR, *and others.*

Since $\tan r=n/\sin s=\surd[\sin(s-a)\sin(s-b)\sin(s-c)]/\surd(\sin s)$, we get the sinister

$$=2\sin s+2\sin(s-a)\sin(s-b)\sin(s-c)$$

$$=2\sin s+\sin(s-c)[\cos(a-b)-\cos c]$$

$$=2\sin s+\tfrac{1}{2}\sin(s-c+a-b)+\tfrac{1}{2}\sin(s-c-a+b)-\tfrac{1}{2}\sin s+\tfrac{1}{2}\sin(2c-s)$$

$$=\tfrac{3}{2}\sin s+\tfrac{1}{2}\sin(2a-s)+\tfrac{1}{2}\sin(2b-s)+\tfrac{1}{2}\sin(2c-s)$$

$$=\Sigma\tfrac{1}{2}[\sin s+\sin(2a-s)]=\Sigma\sin a\cos(s-a),$$

which is the dexter.

15692. (Professor SANJÁNA, M.A.)—At every point P of a parabola the radius of curvature, PO, is taken, and from O the remaining normal, OP′, is drawn to the curve. Prove that the envelope of the chord PP′ is a parabola with the same vertex and with its concavity in the opposite direction.

Solution by A. S. TOMBE, M.A., *and* P. V. SESHU.

Let the point P be $(am^2, 2am)$ and P′ be (x_1, y_1). Since the sum of the ordinates of the feet of the normals from a point $=0$, therefore

$$4am+y_1=0,$$

whence P′ given by $(4am^2, -4am)$. The equation of PP′ is therefore

$$4am^2-my-2x=0;$$

therefore the envelope is $\quad y^2=-32ax.$

15667. (Professor NEUBERG.)—Chercher la condition pour que les équations $\tan x = a\tan(y-z)$, $\tan y = b\tan(z-x)$, $\tan z = c\tan(x-y)$ soient compatibles.

Additional Solution by the PROPOSER.

La première équation peut s'écrire

$$\frac{\tan x}{\tan(y-z)} = a, \quad \frac{\tan x+\tan(y-z)}{\tan x-\tan(y-z)} = \frac{a+1}{a-1}, \quad \frac{\sin(x+y-z)}{\sin(x-y+z)} = \frac{a+1}{a-1}.$$

De même $$\frac{\sin(y+z-x)}{\sin(y-z+x)} = \frac{b+1}{b-1}, \quad \frac{\sin(z+x-y)}{\sin(z-x+y)} = \frac{c+1}{c-1};$$

d'où l'on déduit la condition cherchée

$$\frac{a+1}{a-1}\,\frac{b+1}{b-1}\,\frac{c+1}{c-1} = 1 \quad \text{ou} \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{abc} = 0.$$

8874. (Professor GENESE, M.A.)—The locus of the centres of sections of the conicoid $f(xyz) = 0$ by planes containing the axis of z is the conic determined by $df/dz = 0$, $x\,(df/dx)+y\,(df/dy) = 0$.

Additional Solution by Professor NANSON.

The plane section whose centre is the point α, β, γ is

$$(x-\alpha)f_\alpha+(y-\beta)f_\beta+(z-\gamma)f_\gamma = 0,$$

and, if this passes through the axis of z, then $f_\gamma = 0$ and $\alpha f_\alpha+\beta f_\beta = 0$, whence the result stated. (*Cf. Reprint*, New Series, Vol. II., p. 36.)

15697. (Professor E. B. ESCOTT.)—Find all the integral solutions, if possible, of the equation $x^2-17 = y^3$.

Solutions (I.) *by* R. F. DAVIS, M.A.; (II.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.

(I.) In other words, we require to find integral values of y such that $y^3+17 = \square$. Let $y = t-1$; then $t^3-3t^2+3t+16 = \square = (\lambda t+4)^2$, say; whence $t^2-(\lambda^2+3)\,t+3-8\lambda = 0$. This equation is a quadratic in t, if λ be known; or it is a quadratic in λ, if t be known. Now, if one root of a quadratic with rational coefficients be rational, so also is the other root. Supposing $t = \alpha$, $\lambda = \beta$ satisfy the above equation, if we substitute α for t, we get (solving for λ) $\lambda = \beta$ or $\lambda = \gamma$; then, substituting γ for λ, we get (solving for t) $t = \alpha$ and $t = \delta$; and so on. Hence any single solution leads to an interminable chain of other solutions. (See my note in *Proc. Edin. Math. Soc.*, Vol. XIII., p. 179.) It will be found that $t = -1, 0, 3, 5, 9, 53, 5235$. Thus $y = -2, -1, 2, 4, 8, 52, 5234, \ldots$

$$[5234^3+17 = 14{,}33841{,}52921 = 378661^2].$$

(II.) A really general solution of this Question is difficult to find. But the linear forms of y to modulus 10, 10^2, 10^3 can be readily found by considering the possible endings of the squares (x^2). From the possible units figures of squares, it follows at once that

$$y = 10\eta + 2,\ 3,\ 4,\ 7,\ 8,\ 9, \quad \text{and} \quad y \neq 10\eta + 0,\ 1,\ 5,\ 6\,;$$

With these forms of y, and examining the possible two-figure endings of squares, and using the abbreviations ω = any *odd* digit, ϵ = *any even* digit, a = *any* digit, it follows that the *only* forms of y are

$$y = 100\eta + 02,\ 52\,;\ \epsilon 3\,;\ a4\,;\ 27\,;\ a8\,;\ \omega 9.$$

Again, with these forms of y, and examining the three-figure-endings of squares, it will be found that the only forms of y are

$$y = 1000\eta + 002,\ 102,\ 502,\ 602,\ 902\,;\ 052,\ 252,\ 352,\ 752,\ 852\,;$$
$$127,\ 227,\ 427,\ 627,\ 727,\ 927\,;$$
$$a\epsilon 3\,;\quad aa4\,;\quad aa8\,;\quad a\omega 9.$$

With the above results, it will be found that the only solution with $y < 500$ are $(x, y) = (2, 5), (4, 9), (8, 23), (43, 282), (52, 375)$.

When $y > 464$, a five-figure table of squares is required to examine the "trial" values of x^2.

15705. (S. C. Bagchi, B.A.) — Four pairs of inverse points are taken on a cubic which is its own inverse in normal co-ordinates. The joins of corresponding points cut a series of straight lines in points ${}_rP_s$ ($r = 1$, $s = 1, 2, 3, 4$ for the first line of the series; $r = 2$, $s = 1, 2, 3, 4$ for the second; and so on). These points are mapped into curves in another part of the plane. The scheme of transformation

$${}_rP_s = \phi\,(x, y, {}_r\lambda_s)$$

gives that the range formed by the points where a parallel to the y-axis in the transformed figure cuts a group of four curves is equi-cross with any of the ranges in the first figure. Show that $\phi = u$ (u being a solution of Riccati's equation) is a possible form.

[*Note.*—The word "inverse" is to be taken in the general sense given by Salmon; see *Higher Plane Curves.*]

Solution by the Proposer.

The equation to the cubic (taking the general sense of "inverse," see Salmon, *Higher Plane Curves*, p. 157, French Edition) may be written $\Delta = 0$, where $\Delta = \begin{vmatrix} a, & b, & c \\ x, & y, & z \\ x^{-1}, & y^{-1}, & z^{-1} \end{vmatrix}$. Any line II′, where I, I′ are corresponding points on $\Delta = 0$, is given by $\Delta_1 = 0$ where

$$\Delta_1 = \begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ \alpha^{-1}, & \beta^{-1}, & \gamma^{-1} \end{vmatrix}$$

[because, if I be (α, β, γ), I′ must be $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$]. Therefore any

line such as II′ passes through a fixed point, viz., (a, b, c). Therefore the range ${}_r\mathrm{P}_s$ has a constant cross-ratio. Now it is easily seen that u, the solution of Riccati's equation, satisfies $\lambda(\mathrm{D}u - \mathrm{B}) = (\mathrm{A} - \mathrm{C}u)$, where A, B, C, D are functions of x and λ is a constant, and u satisfies

$$p + \psi_1 + y\chi_1 + y^2\phi_1 = 0,$$

where $p = dy/dx$ and ψ_1, χ_1, ϕ_1 are functions of x. Therefore, taking four particular solutions corresponding to four values of λ, we see that, if a parallel to y-axis cuts this family of curves, viz.,

$$\phi(x, y, {}_r\lambda_s) = 0, \quad r = 1, 2, 3, 4,$$

in P, Q, R, S, $\qquad (\mathrm{PQRS}) = {}_r\mathrm{P}_s = \text{constant}.$

15691. (James Blaikie, M.A.)—BAC is an angle in a circle, and AB, AC meet a diameter in D and E; D′ and E′ are the images in O of D and E. Prove that BE′, CD′ meet (in A′) on the circumference.

Solution by Professor Sanjána, M.A., *and the late* R. Tucker, M.A.

Let X, Y be the extremities of the diameter passing through the centre O; then XD = YD′ and XD′ = YD; also XE = YE′ and YE = XE′. Hence XD . YE : XE . YD = XE′ . YD′ : XD′ . YE′, *i.e.*, (XY, DE) = (XY, E′D′) and A (XY, DE) = A′ (XY, E′D′). Therefore A (XBCY) = A′ (XBCY); also X, B, C, Y are four points on the circle and A is on the circle; therefore, finally, A′ is also on the circle.

Note.—Let BC meet XY in F, and let F′ be isotomically conjugate to F in XY; then BE′, CD′ AF′ meet in a point. (See Question 14233, Vol. lxxv.) We have thus the following theorem:—XY is a diameter of the circum-circle of ABC; A′, B′, C′ are the isotomic conjugates with regard to XY of the intersections of BC, CA, AB with XY: then AA′, BB′, CC′ are concurrent on the circumference.

15624. (D. Biddle.)—In a cubic equation of form $x^3 - qx - r = 0$, r is the product of two primes. Show what numerical value to attach to q in order that the smaller factor of r may be one of the roots; also find the remaining roots, and prove that Cardan's method does not enable us to solve any given equation of the particular sort, unless the larger factor of r exceed the square of half the smaller one.

Solution by Lt.-Col. Allan Cunningham, R.E.

x_1, x_2, x_3 the three roots; $r = p_1 p_2$ (two primes and $p_1 < p_2$). Let $x_1 = p_1$. Then $p_1 p_2 = r = x_1 x_2 x_3$, whence $x_2 x_3 = p_2$ and $x_1 + x_2 + x_3 = 0$,

and $x_2x_3 + x_3x_2 + x_1x_2 = -q$. Hence $p_2 + x_1(x_2+x_3) = -q$, and $p_2 - p_1^2 = -q$ (which gives the value of q).

To find the other roots x_2, x_3: here, since p_1 is a root, $x^3 - qx - r = 0$, and $p_1^3 - qp_1 - r = 0$; therefore $x^3 - p_1^3 - q(x - p_1) = 0$, whence

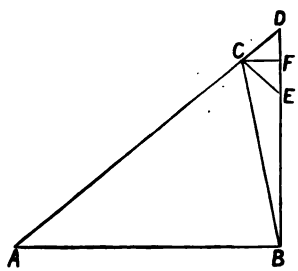
$$x^2 + p_1 x + p_1^2 - q = 0 \quad \text{or} \quad x^2 + p_1 x + p_2 = 0,$$

which gives $x = -\frac{1}{2}p_1 \pm \sqrt{(\frac{1}{4}p_1^2 - p_2)}$ (the required roots).

These two roots will be both real if $p_2 < \frac{1}{4}p_1^2$, and both imaginary if $p_2 > \frac{1}{4}p_1^2$. And Cardan's rule is known to be applicable only when two roots are imaginary, *i.e.*, in the present case when $p_2 > \frac{1}{4}p_1^2$.

15509. (C. M. Ross.)—Having given the base and the altitude of a triangle, and that one of the angles at the base is double the other, show how to construct the triangle.

Another Solution by George Scott, M.A.



Suppose ABC to be the triangle so constructed. Through C draw a parallel to the base. At B, the greater base angle, erect the altitude BF and produce it to meet AC produced in D. Bisect BCF by CE. The angle DCF will be equal to the angle ECF. Call DF = EF z and BF h; let CF = x; let AB = c. Then

$$(h-z)/z = \sqrt{(x^2+h^2)}/x.$$

Again, $c/x = (h+z)/z$; therefore

$$h/z = [\sqrt{(x^2+h^2)}+x]/x.$$

Also $h/z = (c-x)/x$, whence

$$\sqrt{(x^2+h^2)} = c - 2x;$$

therefore $3x^2 - 4cx + c^2 - h^2 = 0$, $\quad x = \frac{1}{3}[2c \pm \sqrt{(c^2+3h^2)}]$.

Mr. Scott discusses in the following manner the problem appended (suggested by Question 15509):—

Given the base and vertical angle of a triangle, to construct it so as to have one base angle double the other.

It is evident that in such a triangle a parallel to the base drawn through the vertex would *trisect* the external vertical angle. Therefore proceed thus:—Lay down the angle supplemental to the given one. Then, by means of the Slide Trammel invented by me, and which I have described in *The Educational Times*, April 1, 1903, trace the curve there mentioned. The instrument should be so adjusted that the *fixed* pivot A should be *in* the line from which the laid down angle is measured, while the slot LM should be perpendicular to this line. Moreover, the distance of A from the vertex of the angle should be double its distance from the slot LM. The point where the curve cuts the second side of the angle should be joined to A. The angle thus formed will be one-third of that laid down. At one end of the given base make an angle equal this third, at the other an angle double it.

9807. (Professor G. B. M. Zerr.)—The perpendiculars from the vertices of a triangle upon the central axis (the line which passes through the circum-centre, the orthocentre, the nine-point centre, and the centroid) are proportional to

$$\cos A \sin(B-C), \quad \cos B \sin(C-A), \quad \cos C \sin(A-B),$$

those on one side of the line being reckoned positive, and those on the other negative.

Solution by the Proposer.

Since the line passes through the centroid and orthocentre, its equation in trilinear co-ordinates is

$$(\sec B \operatorname{cosec} C - \sec C \operatorname{cosec} B)\,\alpha + (\sec C \operatorname{cosec} A - \sec A \operatorname{cosec} C)\,\beta$$
$$+ (\sec A \operatorname{cosec} B - \sec B \operatorname{cosec} A)\,\gamma = 0.$$

This equation immediately follows when we remember that the co-ordinates of centroid and orthocentre are respectively cosec A, cosec B, cosec C, and sec A, sec B, sec C. The co-ordinates of the vertices are respectively $\Delta/(r \sin A)$, 0, 0; 0, $\Delta/(r \sin B)$, 0; 0, 0, $\Delta/(r \sin C)$. The perpendicular distances are given by $(l\alpha' + m\beta' + n\gamma')/\sqrt{(A^2+B^2)}$, where α', β', γ' represent in turn the co-ordinates of the vertices, and l, m, n the coefficients of α, β, γ in the given equation. Therefore the distances are proportional (since Δ/r and $\sqrt{(A^2+B^2)}$ are constant) to

$$\frac{\sin B \cos C - \cos B \sin C}{\sin A \sin B \cos B \sin C \cos C}, \quad \frac{\sin C \cos A - \cos C \sin A}{\sin A \sin B \sin C \cos A \cos C},$$
$$\frac{\sin A \cos B - \cos A \sin B}{\sin A \sin B \sin C \cos A \cos B}.$$

Multiply the numerator and denominator of the first fraction by cos A, of the second by cos B, of the third by cos C; then the denominators of the three fractions are the same and equal to

$$\sin A \sin B \sin C \cos A \cos B \cos C.$$

Hence the perpendiculars are proportional to

$$\cos A \sin(B-C), \quad \cos B \sin(C-A), \quad \cos C \sin(A-B).$$

15664. (R. Chartres.)—Express $1/(r^n+1)^2$ as a radix fraction in the scale radix $= r$.

Solution by the Proposer.

Let $x = \dot{\cdot}0000,\ (r-1)(r-2)\,02,\ (r-1)(r-4)\,04,\ \ldots,\ 01\,(r-1)(\dot{r}-1)$; then

$r^2x = \cdot\dot{0}0\,(r-1)(r-2),\ 02\,(r-1)(r-4),\ 04\,(r-1)(r-6),\ \ldots,\ (r-1)(r-1)\,0\dot{0}$;

therefore $\qquad (r^2+1)\,x = \cdot\dot{0}0\,(r-1)(\dot{r}-1) = 1/(r^2+1)$;

therefore $\qquad x = 1/(r^2+1)^2.$

This supposes r to be odd, but, if r be even, the middle term of the above would be $00\,(r-1)(r-1)$, and the result the same. Similarly for any other value of n.

15646. (R. W. D. Christie.)—Solve the equation $X^2-19Y^2=-3$ (in integers) by the use of other convergents than the ordinary Pellian, and prove its generality: thus

$$\frac{p_n}{q_n}=\frac{2}{1},\ \frac{5}{1},\ \frac{17}{4},\ \frac{22}{5},\ \frac{61}{14}\ \text{(ad inf.)}.$$

Addendum to Solution.

The Proposer desires to add the following to his solution (see *Reprint*, New Series, Vol. viii., p. 28).

Let $p=19$; then the complete cycle of convergents will be as follows (the numbers in parentheses being the multipliers also in cyclical order):—

$$\begin{array}{cccccccccl}
\frac{p_n}{q_n}= & \frac{1}{0}, & \frac{4}{1}, & \frac{9}{2}, & \frac{13}{3}, & \frac{48}{11}, & \frac{61}{14}, & \frac{170}{39} & (1), \ldots, & \text{ad inf.,} \\
 & & (2) & (1) & (3) & (1) & (2) & (8) & & \\
 & \frac{3}{0}, & \frac{2}{1}, & \frac{5}{1}, & \frac{17}{4}, & \frac{22}{5}, & \frac{61}{14}, & \frac{510}{117} & (3), \ldots, & ,, \\
 & & (1) & (3) & (1) & (2) & (8) & (2) & & \\
 & \frac{5}{0}, & \frac{3}{1}, & \frac{14}{3}, & \frac{17}{4}, & \frac{48}{11}, & \frac{401}{92}, & \frac{850}{195} & (5), \ldots, & ,, \\
 & & (3) & (1) & (2) & (8) & (2) & (1) & & \\
 & \frac{2}{0}, & \frac{3}{1}, & \frac{5}{1}, & \frac{13}{3}, & \frac{109}{25}, & \frac{231}{53}, & \frac{340}{78} & (2), \ldots, & ,, \\
 & & (1) & (2) & (8) & (2) & (1) & (3) & & \\
 & \frac{5}{0}, & \frac{2}{1}, & \frac{9}{2}, & \frac{74}{17}, & \frac{157}{36}, & \frac{231}{53}, & \frac{850}{195} & (5), \ldots, & ,, \\
 & & (2) & (8) & (2) & (1) & (3) & (1) & & \\
 & \frac{3}{0}, & \frac{4}{1}, & \frac{35}{8}, & \frac{74}{17}, & \frac{109}{25}, & \frac{401}{92}, & \frac{510}{117} & (3), \ldots, & ,, \\
 & & (8) & (2) & (1) & (3) & (1) & (2) & &
\end{array}$$

and then recur.

Let $\quad a_n=0,\ 4,\ 2,\ 3,\ 3,\ 2,\ 4;\qquad b_n=1,\ 3,\ 5,\ 2,\ 5,\ 3,\ 1;$

$\quad c_n=4,\ 2,\ 1,\ 3,\ 1,\ 2,\ 8;\qquad \ldots\quad \ldots\quad$ ad inf.;

then the general solution is

$$\frac{p_n}{q_n}=\underset{(c_n)}{\frac{b_n}{0}},\quad \underset{(c_{n+1})}{\frac{a_{n+1}}{1}},\quad \underset{(c_{n+2})}{\frac{a_{n+1}c_{n+1}+b_n}{c_{n+1}}}.$$

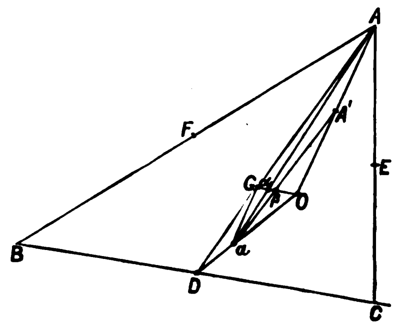
In accordance with the laws of formation,

$$P=a_n^2+b_nb_{n-1},\quad b_nc_{n-1}=a_n+a_{n+1}.$$

When $n=$ half cycle, then a_n or $a_{n+1}=c_n$, and each of them the greatest *odd* root of the prime.

15399. (R. Tucker, M.A.)—ABC is a triangle, O the in-centre; A′, B′, C′ are the mid-points of AO, BO, CO; and a, b, c are the centroids of OBC, OCA, OAB. Show that (i.) $\Delta abc=\frac{1}{9}\Delta ABC$; (ii.) A$a$, B$b$, C$c$ co-intersect in P, and A′a, B′b, C′c co-intersect in Q. Prove that P, Q lie on the join of the in-centre and the centroid of ABC.

Solutions (I.) *by* Professor BROWN, J. BLAIKIE, M.A., *and* T. SAVAGE, B.A. (II.) *by* F. W. REEVES, B.A.; (III.) *by* LIONEL E. REAY, B.A.



(I.) (ii.) Take G the centroid of ABC; then

$$DG = \tfrac{1}{3}DA$$

and $$Da = \tfrac{1}{3}DO;$$

therefore Ga is parallel to OA and $= \tfrac{1}{3}$OA. Let Aa and $A'a$ meet GO in α and β respectively.

$$Ga = \tfrac{1}{3}OA;$$

therefore

$$G\alpha = \tfrac{1}{3}\alpha O;$$

therefore

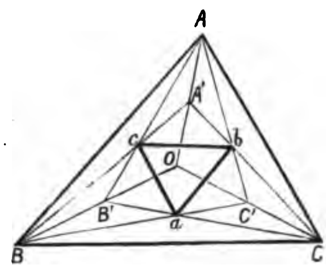
$$G\alpha = \tfrac{1}{4}GO.$$

Similarly it may be proved that Bb meets GO at a point whose distance from G $= \tfrac{1}{4}$GO; also Cc. Therefore Aa, Bb, and Cc meet at a point on GO.

Again $Ga = \tfrac{2}{3}OA'$; therefore $G\beta = \tfrac{2}{3}\beta O$; therefore $G\beta = \tfrac{2}{5}GO$. Similarly it may be proved that $B'b$ meets GO at a point whose distance from G $= \tfrac{2}{5}$GO; also $C'c$. Therefore $A'a$, $B'b$, and $C'c$ meet at a point on GO.

(i.) $Oa = \tfrac{2}{3}OD$ and b lies on OE; so that $Ob = \tfrac{2}{3}OE$; therefore $ab = \tfrac{2}{3}DE = \tfrac{1}{3}AB$. Similarly $bc = \tfrac{1}{3}BC$ and $ac = \tfrac{1}{3}AC$; therefore

$$\Delta abc = \tfrac{1}{9}\Delta ABC.$$



(II.) Now a, b are the points of trisection of BC′, C′A; therefore ab is parallel to AB and $= \tfrac{1}{3}$AB. Similarly for other sides of the triangle abc. Therefore

(i.) $\Delta abc = \tfrac{1}{9}\Delta ABC$,

to which it is similar.

Again, because a, b are the points of trisection on the sides BC′, C′A of the triangle ABC′, Aa, Bb intersect in P; so that $aP = \tfrac{1}{4}Aa$. Similarly Aa, Cc intersect at P; therefore Aa, Bb, Cc meet at the point P of quadrisection.

Further, the corresponding sides of the triangles A′B′C′, abc meet in pairs on the line at infinity and therefore $A'a$, $B'b$, $C'c$ meet at Q. Now the three triangles ABC, A′B′C′, abc have as common axis of perspective the line at infinity; therefore their centres of perspective taken two and two are collinear, *i.e.*, OPQ is a straight line. Also, if G is the centroid of the triangle ABC and D, E, F are the mid-points of BC, CA, AB, the triangle DEF has its sides corresponding to those of the triangle ABC parallel to them, *i.e.*, meeting at infinity, and G is

the centre of perspective of DEF, ABC. Hence OGP is a straight line, *i.e.*, P, Q lie on the join of in-centre and centroid of ABC.

(III.) (i.) The figure OA'*b*C' (see figure of Solution II.) is $\frac{1}{3}\Delta$AOC, for A*b*C = $\frac{1}{3}$, *b*CC' = $\frac{1}{6}$, ...; therefore the whole figure

$$A'bC'aB'c = \tfrac{1}{3}\Delta ABC \quad\ldots\ldots\ldots\ldots(a).$$

Again, $\quad \Delta C'ab = \frac{1}{9}\Delta AC'B = \frac{1}{9}(\Delta ABC - \Delta AC'C - \Delta BB'C)$

$\qquad\qquad = \frac{1}{9}(\Delta ABC - \frac{1}{3}\Delta AOC - \frac{1}{3}\Delta BOC)$;

therefore $\Sigma\Delta C'ab = \frac{1}{9}(3\Delta ABC - \Sigma\Delta AOC) = \frac{2}{9}\Delta ABC \ldots\ldots\ldots\ldots(b)$;

therefore, from (a) and (b), $\Delta abc = \frac{1}{9}\Delta ABC$.

(ii.) If G is the centroid of the triangle ABC, then AG and O*a* both pass through D, the mid-point of BC. Then *a*G = $\frac{1}{3}$AO and is parallel to it. Hence OG and A*a* meet at P such that GP = $\frac{1}{3}$PO; therefore OP = $\frac{3}{4}$OG. Similarly B*b* and C*c* pass through the same point P on OG.

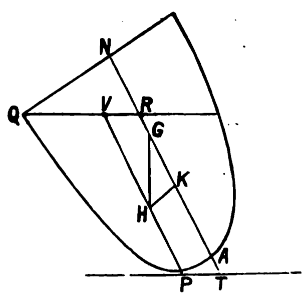
(iii.) Again, since OA' = $\frac{1}{3}$OA, O*a* = $\frac{2}{3}$OD, then A'*a* cuts OG at Q, such that OQ = $\frac{3}{5}$OG. Similarly B'*b*, C'*c* pass through the same point.

10872. (Professor Hudson, M.A.)—A paraboloid of revolution floats with the lowest point of its base in the surface of a fluid, and its axis inclined at an angle θ to the horizon. Find its height and specific gravity.

Note by the Editor.

This problem is identical with Question 6361, which was solved by the Proposer in the *Reprint*, Vol. LXXIII., p. 73. We insert as interesting the alternative solution just furnished by him.

Solution by the Proposer.



Consider a section of the paraboloid by a plane through the axis and the lowest point of the base. Let A be the vertex, G the centre of gravity, H the centre of gravity of the fluid displaced, Q the lowest point of the base.

Let TARN be the axis, PT the horizontal tangent. Draw QVR parallel to PT, PV parallel to AN, HK perpendicular to AN.

Now $\quad NR = 2\cot\theta\sqrt{(ah)}$,

$\qquad\qquad AT = a\cot^2\theta$;

therefore

$$PV = TR = h + a\cot^2\theta - 2\cot\theta\sqrt{(ah)} = (\sqrt{h} - \cot\theta\sqrt{a})^2;$$

also AG = $\frac{2}{3}$AN, PH = $\frac{2}{3}$PV, GH is vertical; therefore

$$KG = KH\tan\theta;$$

therefore $\frac{2}{3}h - a\cot^2\theta - \frac{2}{3}[h + a\cot^2\theta - 2\cot\theta\sqrt{(ah)}] = 2a\cot\theta - \tan\theta$

whence $$h = a[\tfrac{1}{4}(5\cot\theta + 6\tan\theta)]^2.$$

Specific gravity $= 2a\text{PV}^2/2ah^2 = (1 - \cot\theta\sqrt{a/h})^4$

$= [(1 + 5\sin^2\theta)/(5 + \sin^2\theta)]^4.$

6723. (C. Leudesdorf, M.A.)—A pair of tangents to a given conic form a harmonic pencil with two straight lines whose directions are given and which include a right angle. Show that the locus of the point of intersection of the tangents is a rectangular hyperbola, except in the case where the given conic is a parabola, when the locus is a straight line.

Solution by Professor Nanson.

If the tangents from P to two conics Σ, Σ' form a harmonic pencil, the locus of P is a conic S through the points of contact of the common tangents to Σ, Σ'. Now let Σ' reduce to two points G, G'. Thus, if the tangents from P to Σ are harmonically conjugate to PG, PG', the locus of P is a conic through G, G' and through the points of contact of the tangents from G, G' to Σ. But, if Σ touches GG' at A, the locus S of P breaks up into the line GG' and a line cutting GG' in the harmonic conjugate of A. Taking G, G' to be harmonically conjugate with respect to the focoids, we get the results in the Question, and the conic S passes through the four foci of Σ. Thus the rectangular hyperbola is concentric with Σ and is unchanged if Σ be replaced by any confocal. Also, when Σ is a parabola, the finite locus of P is a line through the focus of Σ harmonically conjugate to the axis of Σ with respect to the fixed directions, and is therefore unchanged if Σ be replaced by a confocal parabola.

For other solutions of this Question see Vol. LXXV., pp. 58, 94.

15617. (A. M. Nesbitt, M.A.)—(Suggested by Question 15492.)—An equilateral hyperbola makes intercepts a, b on the x axis and c, d on the y axis (axes rectangular). Prove that (1) $ab + cd = 0$, and that (2) the normals at the four points will be concurrent if

$$a^2 + b^2 = c^2 + d^2 + ab.$$

Solution by C. M. Ross.

(1) The equation $lx^2 + 2mxy - ly^2 + 2px + 2qy + r = 0$ ....(1)

represents an equilateral hyperbola. The intercepts made by (1) on the x axis are given by $lx^2 + 2px + r = 0$ ....(2),

and the intercepts on the y axis are given by

$$ly^2 - 2qy - r = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Now a and b are the roots of (2) and c and d those of (3); therefore

$$ab + cd = r/l - r/l = 0.$$

(2) The equation of any conic passing through the intersections of (1) with the x and y axes is of the form

$$lx^2 + 2mxy - ly^2 + 2px + 2qy + r + \lambda xy = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The equation of the normal at $x'y'$ to (1) is

$$x(mx' - ly' + q) - y(lx' + my' + p) - x'(mx' - ly' + q) + y'(lx' + my' + p) = 0.$$

If this normal pass through the point (h, k),

$$h(mx' - ly' + q) - k(lx' + my' + p) - x'(mx' - ly' + q) + y'(lx' + my' + p) = 0.$$

This shows that the point (x', y') lies on the equilateral hyperbola

$$mx^2 - 2lxy - my^2 - x(mh - lk - q) + y(l + mk - p) + pk - qh = 0.$$

Now this conic passes through the same four points as does (4)...(5). Comparing (4), (5),

$$l = m, \quad 2m + \lambda = -2l, \quad 2p = -mh + lk + q, \quad 2q = lh + mk - p, \quad r = pk - qh.$$

Also, from equations (2) and (3),

$$a + b = -2p/l, \quad ab = r/l, \quad c + d = 2q/l, \quad cd = -r/l.$$

Again $\quad mh - lk + 2p - q = 0, \quad lh + mk - (p + 2q) = 0, \quad qh - pk + r = 0.$

Eliminating h and k from these equations,

$$\begin{vmatrix} m & -l & 2p - q \\ l & m & -(p + 2q) \\ q & -p & r \end{vmatrix} = 0;$$

therefore $2lr - 3p^2 + 3q^2 - 2pq = 0$ on reduction; therefore

$$2r/l - 3(p/l)^2 + 3(q/l)^2 - 2(p/l)(q/l) = 0;$$

therefore $\qquad 3(a^2 + b^2) = 3(c^2 + d^2) - 4ab + 2(a + b)(c + d),$

the condition for concurrency.

N.B.—I fail to obtain Mr. Nesbitt's relation.

15666. (J. J. Barniville, B.A., I.C.S.)—Having $u_n + u_{n+1} = u_{n+4}$, prove that

$$\frac{1}{3.4.5.6} + \frac{1}{3.5.6.7} + \frac{1}{4.6.7.9} + \frac{1}{5.7.9.11} + \ldots = \frac{1}{180},$$

$$\frac{1.3.10}{3.4.6.7} + \frac{2.4.12}{3.5.7.9} + \frac{2\,..15}{4.6.9.11} + \frac{2.6.18}{5.7.11.13} + \ldots = \frac{1}{2}$$

Solution by Professor SANJÁNA, M.A., *and* C. M. ROSS.

The following series,

$$n = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ \ldots,$$
$$u_n = 3,\ 3,\ 4,\ 5,\ 6,\ 7,\ 9,\ 11,\ 13,\ 16,\ 20,\ \ldots,$$

satisfies the given relation. Now the $(n+1)$-th term of the first sum

$$= \frac{1}{u_n u_{n+2} u_{n+3} u_{n+4}} = \frac{u_{n+1}}{u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4}} = \frac{u_{n+4}-u_n}{u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4}}$$
$$= \frac{1}{u_n u_{n+1} u_{n+2} u_{n+3}} - \frac{1}{u_{n+1} u_{n+2} u_{n+3} u_{n+4}}.$$

Hence the sum to infinity, as the series is convergent, is

$$\frac{1}{u_0 u_1 u_2 u_3} = \frac{1}{3\,.\,3\,.\,4\,.\,5} = \frac{1}{180}.$$

The series 1, 2, 2, 2, 3, 4, 4, 5, 7, 8, 9, 12, ..., which follows the same law, has its terms respectively equal to u_2-u_0, u_3-u_1, u_4-u_2, u_5-u_3, u_6-u_4, ... ; and 10, 12, 15, 18, 22, 27, ... has the respective terms u_2+u_4, u_3+u_5, u_4+u_6, u_5+u_7, ... Therefore the $(n+1)$-th term of the second series

$$= \frac{(u_{n+2}-u_n)\,u_{n+1}\,(u_{n+2}+u_{n+4})}{u_n u_{n+2} u_{n+4} u_{n+5}} = \frac{u_{n+1}\,(u_{n+2}^2 + u_{n+2}u_{n+4} - u_n u_{n+2} - u_n u_{n+4})}{u_n u_{n+2} u_{n+4} u_{n+5}}$$

$$= \frac{u_{n+1}\,(u_{n+2}^2 + u_{n+2}u_{n+1} - u_n u_{n+4})}{u_n u_{n+2} u_{n+4} u_{n+5}} = \frac{u_{n+1}\,(u_{n+2}u_{n+5} - u_n u_{n+4})}{u_n u_{n+2} u_{n+4} u_{n+5}}$$

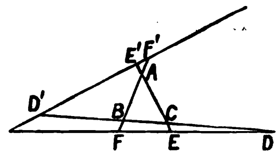
$$= \frac{u_{n+1}}{u_n u_{n+4}} - \frac{u_{n+1}}{u_{n+2} u_{n+5}} = \frac{1}{u_n} - \frac{1}{u_{n+4}} - \frac{1}{u_{n+2}} + \frac{1}{u_{n+5}}.$$

Applying this mode of decomposition to every term, adding up and cancelling unlike equal terms, we get for the sum

$$\frac{1}{u_0} - \frac{1}{u_4} + \frac{1}{u_1} = \frac{1}{3} - \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$

15673. (JAMES BLAIKIE, M.A.)—A straight line meets BC, CA, AB, the sides of a triangle ABC, in D, E, F, and CB is produced to D′, so that BD′ = DC ; CA is produced to E′, so that AE′ = EC ; BA is produced to F′, so that AF′ = FB. Prove that D′, E′, F′ are collinear without assuming any property of the hyperbola.

Solution by the PROPOSER *and others.*



By Menelaus' theorem

$$BD \cdot CE \cdot AF = CD \cdot AE \cdot BF.$$

It follows that

$$CD' \cdot AE' \cdot BF' = BD' \cdot CE' \cdot AF';$$

therefore, by the converse of Menelaus' theorem, D', E', F' are collinear.

15698. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Factorize into *prime* factors $N = (2^{127}+2^{63}+1)^2+2^{64}$; this contains 77 figures.

Solutions (I.) *by the* PROPOSER; (II.) *by* JAMES BLAIKIE, M.A.

(I.) The method of factorizing numbers of this kind is contained in the writer's paper on "High Primes, $p = 4\varpi+1$, $6\varpi+1$, and Factorisations," Arts. 11, 12, in *Quarterly Journal of Pure and Applied Mathematics*, Vol. xxxv., 1903. It is shown (Art. 12) that, if

$$(N) = N_{p-y}N_{p+y},$$

where $Ny = y^2+1 = p$ (for shortness), and note that p need not here be *prime* (although used to denote a prime in the paper), then

$$(N) = \{(p-y)^2+1\}\{(p+y)^2+1\} = (p^2-y^2+1)^2+(2y)^2$$
$$= (y^4+y^2+2)^2+(2y)^2.$$

Also $$N_{p\mp y} = (p\mp y)^2+1 = (y^2\mp y+1)^2+1^2.$$

Now, take $y = 2^{32}$; then $\frac{1}{4}(N) =$ the proposed number N, and the complete factorization of N_{p-y} and N_{p+y}, when $y = 2^{32}$, has been given in the writer's solution of Question 15608.

N.B.—This is the highest number of this kind at present factorizable; it involves $(2^{64}+1)$ and $(2^{126}+1)$.

(II.) $$N = (2^{127}+2^{63}+1)^2+2^{64} = 2^{254}+2^{191}+2^{128}+2^{126}+2^{65}+1$$
$$= (2^{126}+1)(2^{128}+2^{65}+1) = 2^{126+1}(2^{64}+1)^2.$$

Now

$$2^{126}+1 = (2^{63}+2^{32}+1)(2^{63}-2^{32}+1) = (2^{43}+1)(2^{86}-2^{43}+1)$$
$$= (2^{21}-2^{11}+1)(2^{21}+2^{11}+1)(2^{42}+2^{32}+2^{21}+2^{11}+1)(2^{42}-2^{32}+2^{22}-2^{11}+1).$$

Also $$2^{21}-2^{11}+1 = (2^7+2^4+1)(2^{14}-2^{11}+2^7-2^4+1)$$
$$= (2^2+1)(2^5-2^2+1)(2^{14}-2^{11}+2^7-2^4+1),$$

and $$2^{21}+2^{11}+1 = (2^7-2^4+1)(2^{14}+2^{11}+2^7+2^4+1)$$
$$= (2^7-2^4+1)(2^4-2^2+1)(2^{10}+2^8+2^7+2^4+2^2+1).$$

Substituting numerical values, we find

$$2^{126}+1 = 5.13.29.113.1429.1449.4393753638913.4402343577601.$$

But $2^{126}+1 = (2^{18}+1)(2^{108}-2^{90}+\ldots+1)$ and

$$2^{18}+1 = (2^9+2^5+1)(2^9-2^5+1)$$
$$= (2^2+1)(2^4-2^2+1)(2^5+2^2+1)(2^7-2^4-2^2+1) = 5.13.37.109.$$

Thus we see that 37 and 109 are factors of $2^{126}+1$. Dividing the last two factors by them, we finally obtain

$$N = 5.13.29.37.109.113.14449.40388473189.118750098349$$
$$.(18446744073709551617)^2.$$

By M. Landry's result, quoted by Col. Cunningham in Question 15608, we have $$2^{64}+1 = 274177 . 67280421310721;$$ therefore we may substitute for the final squared factor of N $$(274177)^2 . (67280421310721)^2.$$

15711. (I. Arnold.)—The sides of a plane triangle are in arithmetical progression. It is required to construct it when the common difference and vertical angle are given.

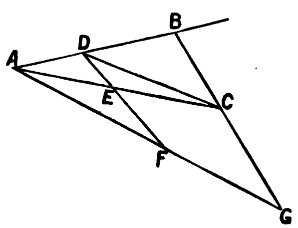
Solution by the late R. Tucker, M.A., *and* C. M. Ross.

(1) When the vertical angle is opposite the mean side c and δ and A are given, $$\cos A = [(c-\delta)^2+(c+\delta)^2-c^2]/[2(c^2-\delta^2)];$$ therefore $$c^2 = 2(1+\cos A)\,\delta^2/(2\cos A-1);$$ therefore, &c.

(2) When the vertical angle is opposite the shortest side. With the same notation,
$$\cos A = [(c+\delta)^2+c^2-(c-\delta)^2]/[2c(c+\delta)] = (c+4\delta)/[2(c+\delta)];$$
therefore $$c = 2\delta(2-\cos A)/(2\cos A-1).$$

(3) The case where the vertical angle is opposite the greatest side may be similarly considered.

The Proposer gives the following interesting geometrical solution of case (3):—



Let the longest side be the base of the triangle. In the indefinite line AB take AD = the common difference. Make the angle ADF equal to the given vertical angle, and take DF = 2AD. Join AF and make $\angle$ EAF = $\angle$ AFE. Bisect the angle BDF by DC, meeting AE produced in C. Through C draw CB parallel to DF; then is ABC the required triangle. Produce AF and BC to meet in G. Then the triangles ADF, ABG are similar, and DF = 2AD; therefore BG = 2AB. Because $\angle$ FAE = $\angle$ AFE = $\angle$ AGC, then AC = CG, and AC + CB = BG = 2AB, or AB is an arithmetic mean between AC and CB. Again, since $\angle$ BDC = $\angle$ CDE = $\angle$ DCB, therefore BD = BC and AD is the common difference. Also $\angle$ ABC = $\angle$ ADF, the given angle. Therefore, &c.

8108. (B. Hanumanta Rau, M.A.)—Two knights being placed on two squares of a chess-board, required to move each 31 times so that no square may be used more than once.

Solution by T. Dennis.

Let P and Q be the two knights. The method of solution will be as follows :—When P and Q are given in any arbitrary positions, we shall

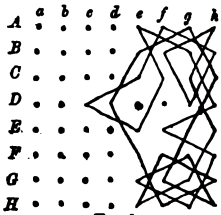


Fig. 1.

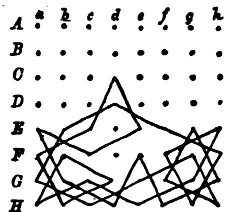


Fig. 2.

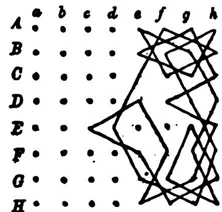


Fig. 3.

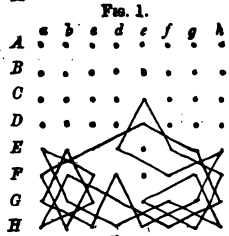


Fig. 4.

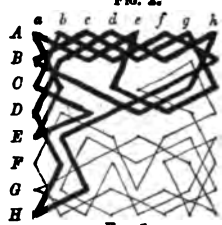


Fig. 5.

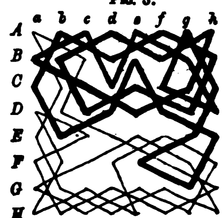


Fig. 6.

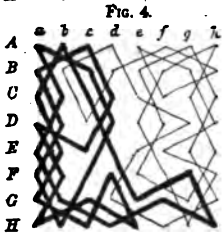


Fig. 7.

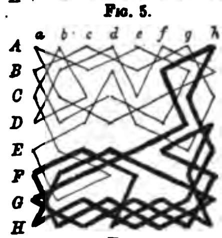


Fig. 8.

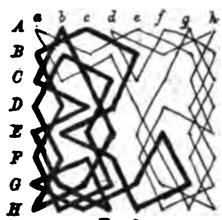


Fig. 9.

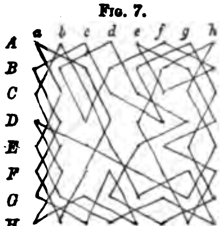


Fig. 10.

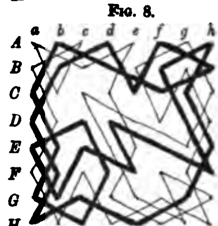


Fig. 11.

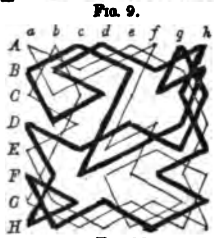


Fig. 12.

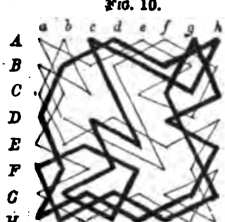


Fig. 13

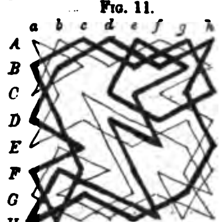


Fig. 14

endeavour to find two closed paths made up of knights' moves, one path

passing through P and not Q and the other passing through Q and not P. If each of these paths consists of 32 squares and no square is common to the two paths, then we have found a solution for the position considered. Also, since we shall only consider closed paths, this solution also applies if P and Q are initially anywhere on their respective paths. If initially P and Q are a knight's move off one another, then a single path completing the whole board will solve. We use this in Fig. 10. Figs. 1, 5, 10, and 11 are fundamental; the others are derived from these by reflexion in a diagonal or by twisting the board round. Since Fig. 1 is symmetrical about a line bisecting the board and parallel to the edge ABC ... H, we need not draw in the other path in this case.

Now place P on the board; then turn the board round so that P is on the lower right quarter of the board (for brevity we shall call this quarter X).

(α) Consider Figs. 1 and 2: the only squares that are on the P path (shown by a fine line) in both figures are those in X and the square D*d* and D*d* is not on the P path in Fig. 3. Hence one of these three figures solves the problem, unless P and Q are both on X.

(β) Let Q be on E*e*. Fig. 3 solves unless P is on E*f*. So we want a figure in which E*e* and E*f* are on different paths; Fig. 4 is such, and hence it solves this case.

(γ) Similarly, if Q be on E*f*, Fig. 3 solves unless P is on E*e*, in which case Fig. 4 solves.

(δ) Now let Q be on E*g*: Fig. 8 solves.

(ϵ) Now let Q be on E*h*: Fig. 6 solves unless P is on E*g*, F*e*, or G*g* these cases are solved by Figs. 8, 4, and 9 respectively.

(ζ) Next let Q be on F*f*: Fig. 12 solves unless P is on E*e*, E*g*, E*h*, G*h*, H*f*. We have just considered the first three of these cases (with P and Q interchanged); the other two are solved by Fig. 13.

(η) *Case of failure.*—Now let Q be on F*g*: Fig. 7 solves unless P is on G*e*, G*f*, H*e*, or H*h*; the first case is solved by 9. The second case is one of failure. If the two knights are placed on any of the four pairs of squares which are a knight's move from a corner, then a solution is impossible. For it is impossible for a path to get into the corner and out again. If P is on H*e*, Fig. 9 solves; if P is on H*h*, Fig. 10 solves.

(ξ) Now let Q be on F*h*: Fig. 11 solves unless P is on E*f*, E*h*, F*e*, G*g*, or H*e*; the first three of these have been considered; the others are solved by Fig. 9.

(θ) Next let Q be on G*g*: Fig. 9 solves unless P is on E*f*, G*e*, or H*e*; the first of these cases has been considered; the others are solved by Figs. 7 and 6 respectively.

(ϕ) Now let Q be on G*h*: Fig. 12 solves unless P is on E*e*, E*g*, E*h*, F*f*, H*f*; all these have been considered except H*f*, which is solved by Fig. 14.

(ψ) Next let Q be on H*h*: Fig. 7 solves unless P is on F*g*, G*e*, G*f*, H*e*. The first case has been considered. The others are solved by Figs. 9, 10, 13 respectively. We have now shown that we may regard P as on

one of sixteen of the squares and have considered the problem for the cases of Q on any of the dotted squares on the adjoining diagram

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
A	·	·	·	·	·	·	·	·
B	·	·	·	·	·	·	·	·
C	·	·	·	·	·	·	·	·
D	·	·	·	·	·	·	·	·
E	·	·	·	·	·	·	·	·
F	·	·	·	·		·	·	·
G	·	·	·	·			·	·
H	·	·	·	·				·

(χ) Now interchange great and small letters in the last part of the work, and we get solutions for the remaining squares.

We append a diagram for reference. When Q is placed initially on any square, the Greek letter in that square refers to the part of the work where the case of Q on this square (and P anywhere on X) is considered.

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*
A	α	α	α	α	α	α	α	α
B	α	α	α	α	α	α	α	α
C	α	α	α	α	α	α	α	α
D	α	α	α	α	α	α	α	α
E	α	α	α	α	β	γ	δ	ϵ
F	α	α	α	α	χ	ζ	η	ξ
G	α	α	α	α	χ	χ	θ	ϕ
H	α	α	α	α	χ	χ	χ	ψ

15715. (Professor E. B. Escott.) — In Fermat's (Pell's) equation $x^2 - Ny^2 = 1$, where N is a prime of the form $4n+3$, prove that the middle partial quotient of $\sqrt{N}$ expressed as a continued fraction is always odd and equal to a or $a-1$ according as a is odd or even (a being

the integral part of $\sqrt{N}$). In the last case the quotient immediately preceding the middle quotient is unity.

Solution by ALEXANDER HOLM, M.A.

If the number of terms in the cycle of partial quotients were odd, then the number N would be the sum of two integral squares: *vide The Expression of a Quadratic Surd as a Continued Fraction*, by Thomas Muir, Glasgow (Maclehose), 1874, § 51. But a number of the form $4n+3$ cannot be the sum of two squares. Therefore the number of terms in the cycle must be even. Now "when the number N is a prime and there is an even number of terms in the cycle of partial quotients then the middle term is a or $a-1$ according as a is odd or even" (Muir, *op. cit.*, § 47). The properties of continued fractions of this kind are very fully investigated in a paper "On the Phenomenon of Greatest Middle in the Cycle of a Class of Periodic Continued Fractions," by Thomas Muir, *Proc. Roy. Soc. Edin.*, 1883–84. In § 18 it is proved that when a is even the quotient immediately preceding the middle quotient is unity.

15717. (R. CHARTRES.)—Find integral values of x, and n ($n>3$), so that x^n-1 shall equal the product of two consecutive integers. When $n=3$,

$$7^3-1=342=18\cdot 19.$$

Note by Lt.-Col. ALLAN CUNNINGHAM, R.E.

This seems a difficult problem. If $a(a+1)$ be the product in question, then $x^n=a^2+a+1=(a^3-1)/(a-1)$; therefore x must be odd, and of form (A^2+3B^2). Hence x must be either a prime of form $(6\varpi+1)$, or a power of such a prime; or else a product of such primes, or of powers of such primes. Again, $\frac{1}{2}(x^n-1)=\frac{1}{2}a(a+1)$, the triangular number of base a. On comparing the values of $\frac{1}{2}(x^n-1)$, for the possible forms of x, with Joncourt's *Table of Triangular Numbers*, it is found that no solution exists when $x^n<2.10^8$, and $n>3$ (as required). Note that the solution of $(a^3-1)/(a-1)\equiv 0 \pmod{x^n}$ is in itself a difficult enough problem (when $n>3$); but the present problem is much more difficult.

Mr. R. W. D. CHRISTIE contributes the following discussion:—

In attempting to solve Question 15717 I came to the conclusion that integral solutions were impossible from the general solution except for the one value $a=2$, *e.g.*, let $x^n-1=a(a+1)=a^2+a$. Thus

$$x^n=a^2+a+1.$$

It follows that x is of the same form; therefore $(a^2+a+1)^n=A^2+A+1$. For $n=1$ the solution is obvious, $a=A$. For $n=2$ we have

$$(a^2+a+1)^2=(a^2-1)^2+(a^2-1)(2a+1)+(2a+1)^2.$$

For $n=3$

$$(a^2+a+1)^3=(a^3-3a-1)^2+(a^3-3a-1)[3a(a+1)]+[3a(a+1)]^2$$

where $a^3-3a-1=1$ necessarily; thus $a=2$, a unique solution.

For $n = 4$ $\qquad (a^2+a+1)^4 = (a^4-6a^2-4a)^2+\dots,$

where $\qquad a^4-6a^2-4a = 1$ or $4a^3+6a^2-1 = 1$,

both impossible. For $n = 5$

$$(a^2+a+1)^5 = (a^5-10a^3-10a^2+1)+\dots.$$

Here $a^5-10a^3-10a^2+1 = 1$, impossible. The general solution is Legendre's; *vide* it in Euler's *Algebra*, p. 586, in an extended form. We must take on the dexter the alternate coefficients of $(a+b)^n = A$, giving form (A^2+AB+B^2), and then equate either A or B to unity in order to secure A^2+A+1 or B^2+B+1.

I should like to know whether other integral values than 2 can be got, and so am interested in the Question.

15707. (Professor NEUBERG.)—On joint le sommet A d'une ellipse à un point quelconque M de cette courbe; la perpendiculaire en M sur AM rencontre l'ellipse en un second point N; enfin on achève le rectangle AMNP. Trouver le lieu du point P.

Solution by the late R. TUCKER, M.A., *and* R. F. DAVIS, M.A.

$\angle NAM = \phi$, $\quad \angle PAA' = \theta$.

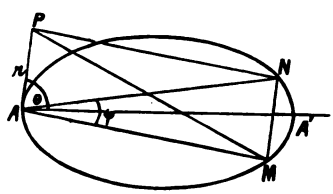
$AN = l$, $\quad r = AP = l\sin\phi$;

co-ordinates of M are

$l\cos\phi\sin\theta$, $\quad -l\cos\phi\cos\theta$;

co-ordinates of N are

$l\sin(\phi+\theta)$, $\quad -l\cos(\phi+\theta)$.



The equation to the ellipse is

$$a^2y^2 = b^2(2ax-x^2).$$

Therefore, since N is on the curve, $\qquad l\left[a^2\cos^2(\phi+\theta)+b^2\sin^2(\phi+\theta)\right] = 2ab^2\sin(\phi+\theta),$

i.e., in co-ordinates $x = r\cos\theta$, $y = r\sin\theta$ of the point P,

$$l\left[(a^2x^2+b^2y^2)\cos^2\phi+(a^2y^2+b^2x^2)\sin^2\phi-2(a^2-b^2)\,xy\sin\phi\cos\phi\right] = 2ab^2r\,(x\sin\phi+y\cos\phi),$$

i.e., using the relation $r/l = \sin\phi$,

$$(a^2x^2+b^2y^2)\cos^2\phi-2y\left[(a^2-b^2)\,x+ab^2\right]\sin\phi\cos\phi + (a^2y^2+b^2x^2-2ab^2x)\sin^2\phi = 0 \;\dots\; \text{(i.)}.$$

Since M is on the the curve, therefore

$$a^2\,l\cos\phi\cos^2\theta = b^2\sin\theta\,(2a-l\cos\phi\sin\theta),$$

i.e., $\qquad l\cos\phi\left[a^2\cos^2\theta+b^2\sin^2\theta\right] = 2ab^2\sin\theta,$

i.e., $\qquad (a^2x^2+b^2y^2)\cos\phi = 2ab^2y\sin\phi \;\dots\dots\dots\dots\; \text{(ii.)}.$

From (i.) and (ii.), by eliminating ϕ, we get

$$4a^2b^4y^2-4ab^2y^2\left[(a^2-b^2)\,x+ab^2\right]+(a^2y^2+b^2x^2-2ab^2x)(a^2x^2+b^2y^2) = 0,$$

i.e., $\qquad -4ab^2\left[(a^2-b^2)\,y^2x\right]+(a^4+b^4)\,x^2y^2+a^2b^2(x^4+y^4) -2ab^2x\,(a^2x^2+b^2y^2) = 0,$

i.e., $\qquad (a^4+b^4)\,x^2y^2+a^2b^2(x^4+y^4)-2ab^2x\left[2y^2(a^2-b^2)+a^2x^2+b^2y^2\right] = 0,$

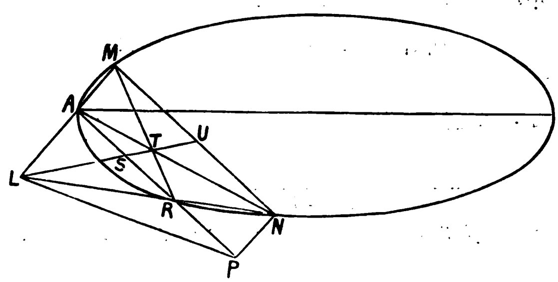
i.e., $\qquad (a^4+b^4)\,x^2y^2+a^2b^2(x^4+y^4)-2ab^2x\left[a^2x^2+(2a^2-b^2)\,y^2\right] = 0,$

i.e., $a^2b^2(x^2+y^2)^2+(a^2-b^2)^2x^2y^2 = 2ab^2x\left[a^2(x^2+y^2)+(a^2-b^2)y^2\right]$,
i.e., in polar co-ordinates

$$r = 2ab^2\cos\theta\,\frac{\left[a^2+(a^2-b^2)\sin^2\theta\right]}{a^2b^2+(a^2-b^2)\sin^2\theta\cos^2\theta}.$$

[Mr. Davis remarks that this is an oval curve lying outside the ellipse and touching it at the extremities of the major axis; also that when $a = b$ it reduces to $x^2+y^2 = 2ax$ as might, *a priori*, be expected.]

Note by Constance I. Marks, B.A.



In connexion with this problem, the following geometrical discussion will perhaps be found interesting, as showing how to determine any number of points on the same locus by means of pairs of perpendicular chords through the vertex A, the rectangles not being completed as in the question.

Let AMNP be the rectangle in the question in any one of its positions, so that P is a point on the locus. Join NA, NR and let MA and NR produced meet in L. Join MR, LP and draw LSTU through T, the intersection of NA and MR. From the harmonic property of the complete quadrilateral formed by the straight lines LM, LN, AN, MR, we have its diagonals AR, MN bisected at S and U; for AR and MN are parallel by hypothesis; therefore the directions AR and LU are conjugate in the ellipse, and LU produced is a diameter. Again, because LA and PN are parallel, therefore the triangles LNA and LPA are equal, whence the triangles ANR, LPR are equal, and have their angles at R equal therefore AR : RP = LR : RN = LA : AM (because AR is parallel to MN).

We have now the following general method for finding positions of P: —Let any perpendicular chords through A (AM, AR, say) be produced through A and R respectively, the former to meet the diameter conjugate to AR in L; then, if on AR produced RP be taken a fourth proportional to LA, AM, and AR, the point P is on the required locus.

15712. (Professor Sanjána, M.A.)—In the triangle ABC, AD is the median to the side BC and GQ is the perpendicular to BC from G, the median point; also AD_1 is the symmedian, and KQ_1 the perpendicular

from K, the symmedian point; segments ER, E_1R_1 and FS, F_1S_1 are similarly taken on CA and AB. Prove that

$$(D_1Q_1/DQ)(b^2+c^2)+(E_1R_1/ER)(c^2+a^2)+(F_1S_1/FS)(a^2+b^2) = 12S\tan\omega.$$

Solution by the late R. TUCKER, M.A., *and others.*

If L is the foot of the perpendicular from A on BC, then DQ/DL = $\frac{1}{3}$;

therefore $\quad DQ = \frac{1}{3}(\frac{1}{2}a - b\cos C) = \frac{1}{6}(c\cos B - b\cos C)$ ....(i.),

$D_1Q_1/D_1L = \frac{1}{2}a\tan\omega/b\sin C$ and $D_1L = [bc/(b^2+c^2)](b\cos B - c\cos C)$;

therefore $\quad D_1Q_1 = \frac{1}{2}a\tan\omega\,\dfrac{bc(b\cos B - c\cos C)}{(b^2+c^2)\,b\sin C}$;

therefore

$$\frac{D_1Q_1}{DQ}(b^2+c^2) = \frac{abc\tan\omega\,(b\cos B - c\cos C)}{2b\sin C\,(c\cos B - b\cos C)}\,6 = \frac{3abc\tan\omega R(\sin 2B - \sin 2C)}{bc\sin(C-B)}$$

$$= \frac{3a\tan\omega\,2R\cos A\sin(C-B)}{\sin(C-B)} = 6aR\tan\omega\cos A;$$

therefore $\Sigma(D_1Q_1/DQ)(b^2+c^2) = 6R^2\tan\omega\,\Sigma\sin 2A = 12S\tan\omega.$

10876. (Professor SYLVESTER.)—If ϕ, ψ, ω are three algebraic functions of x, y, y', y'' such that ϕ', ψ', ω' contain a common factor $\theta(x, y, y', y'', y''')$, show that the complete primitive of $F(\phi, \psi, \omega) = 0$, where F is any function form, may be found algebraically.

[Professor Sylvester remarks that this question is an extension of a well known principle of Lagrange, and that it may itself be indefinitely extended in more than one direction.]

Solution by J. A. H. JOHNSTON, M.A.

From the mode of formation of $\theta(x, y, y', y'', y''')$, it follows that $\phi = a$, $\psi = b$, and $\omega = c$ are three first integrals of $\theta = 0$, where a, b, and c are arbitrary constants.

The stated conditions imply that ϕ, ψ, and ω are independent functions, and, since they involve x, y, y', and y'' algebraically, we may eliminate y' and y'' between $\phi = a$, $\psi = b$, and $\omega = c$.

The result is the complete primitive of $\theta = 0$ or $\chi(x, y, a, b, c) = 0$.

Now, if we differentiate any conceivable functional form $F(\phi, \psi, \omega) = 0$, $(\partial F/\partial\phi)\,\phi' + (\partial F/\partial\psi)\,\psi' + (\partial F/\partial\omega)\,\omega' = 0$ leads again to $\theta = 0$, since it is common to ϕ', ψ', and ω', and so $F(\phi, \psi, \omega) = 0$ is a first integral of $\theta = 0$, but of course non-independently of ϕ, ψ, and ω, which are its three independent first integrals. In fact $F(a, b, c) = 0$. It follows that the complete primitive of $F(\phi, \psi, \omega) = 0$ is $\chi(x, y, a, b, c) = 0$, subject to the condition $F(a, b, c) = 0$, and this reduces the arbitrary constants to two—the proper number.

The solution has thus been found algebraically.

Note.—The multipliers of θ in ϕ', ψ', and ω' may be regarded as integrating factors, and the whole process may be illustrated by the simple case

$$\phi = xy + x^2y' + \tfrac{1}{3}x^3y'' = a, \quad \psi = x^2y + \tfrac{2}{3}x^3y' + \tfrac{1}{3}x^4y'' = b,$$

$$\omega = x^3y + 2x^4y' + \tfrac{1}{3}x^5y'' = c,$$

all leading to a common factor for ϕ', ψ', and ω', viz.,

$$\theta = y + 3xy' + \tfrac{5}{3}x^2y'' + \tfrac{1}{3}x^3y''' = 0.$$

The eliminant of ϕ, ψ, and ω leads to $y = (3ax^2 - 3bx + c)/x^3$ as the complete primitive, and this is easily identified as the complete primitive of $\theta = 0$.

It is to be noted that cases of particular integrals arise from the combined use of $F(\phi, \psi, \omega) = 0$ and the part of its differentiated result that is independent of $\theta = 0$.

The theorem may now be generalized as follows:—Let $\phi_1, \phi_2, \ldots, \phi_n$ be n algebraical functions of x, y, y', y'', ..., $y^{(n-1)}$, such that $\phi'_1, \phi'_2, \ldots, \phi'_n$ contains a common factor $\theta(x, y, y', y'', \ldots, y^{(n)})$. Then the complete primitive of $F(\phi_1, \phi_2, \ldots, \phi_n) = 0$, where F is any function form, is $\chi(x, y, a_1, a_2, \ldots, a_n) = 0$, subject to the condition

$$F(a_1, a_2, \ldots, a_n) = 0,$$

where χ is the eliminant of $\phi_1 = a_1$, $\phi_2 = a_2$, ..., $\phi_n = a_n$.

15699. (James Blaikie, M.A.)—Prove that $m^{2n+1} + (m-1)^{n+2}$ is a multiple of $m^2 - m + 1$ [*e.g.*, $1000^{15} + 999^9 = M(999001)$].

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E., *and the late* R. Tucker, M.A.; (II.) *by the* Proposer; (III.) *by* R. F. Davis, M.A. (IV.) *by* Professor Sanjána, M.A.

(I.) Let $N = m^{2n+1} + (m-1)^{n+2}$, and $q = m^2 - m + 1$. Then

$$N = m(m^2)^n + (m-1)^2(m-1)^n = m(q + m - 1)^n + (m-1)^2.(m-1)^n$$
$$\equiv m(m-1)^n + (m-1)^2.(m-1)^n \pmod q, \equiv (m-1)^n[m + (m-1)^2] \pmod q$$
$$\equiv (m-1)^n.q \equiv 0 \pmod q.$$

(II.)
$$m^{2n+1} + (m-1)^{n+2} = m^{2n-1}(m^2 - m + 1) + (m-1)[m^{2n-1} + (m-1)^{n+1}].$$

If, then, $m^{2n-1} + (m-1)^{n+1}$ be a multiple of $m^2 - m + 1$, then

$$m^{2n+1} + (m-1)^{n+2}$$

is also a multiple, that is, if the proposition be true for one value of n, it is true for all higher values. But it is true when $n = 0$; therefore it is always true.

(III.) Let $u_n = m^{2n+1} + (m-1)^{n+2}$. Then by multiplication it is easily found that $(m^2 + m - 1)u_n = u_{n+1} + m^2(m-1)u_{n-1}$. Hence any factor that is contained in both u_{n-1} and u_n will also be contained in u_{n+1}. But

$$u_0 = m + (m-1)^2 = m^2 - m + 1;$$

and $$u_1 = m^3 + (m-1)^3 = (2m-1)(m^2 - m + 1).$$

Therefore u_2 is a multiple of m^2-m+1; and, since u_1, u_2 are both multiples of m^2-m+1, so also is u_3; and so on.

(IV.) When $m^2-m+1=0$, or m is an imaginary cube root of -1, we can readily show the expression to be zero; hence its divisibility. By the following method, however, we can get an expression for the quotient. Let $m^2-m+1=k$, so that $m-1=m^2-k$; then the given expression

$$= m^{2n+1}+(m^2-k)^{n+2} = m^{2n+1}+m^{2n+4}-C_1m^{2n+2}k+C_2m^{2n}k^2-\dots$$
$$+(-1)^{n+2}k^{n+2}$$
$$= m^{2n+1}(m+1)k-C_1m^{2n+2}k+C_2m^{2n}k^2-\dots$$
$$= k\left[m^{2n+2}+m^{2n+1}-C_1m^{2n+2}+C_2m^{2n}k-\dots\right],$$

where $C_1, C_2, \dots$ are the numbers of 1, 2, ... combinations of $n+2$ things.

15686. (G. H. Hardy, M.A.)—The area Δ and semi-perimeter s of a triangle are fixed. Show that the maximum and minimum values of one of the sides are roots of the equation $sx^2(x-s)+4\Delta^2=0$. Discuss the existence of real maximum and minimum values.

Solution by the Proposer.

If a is stationary, and $a+b+c = 2s = \text{const.}$ .... (i.),

$s(s-a)(s-b)(s-c) = \Delta^2 = \text{const.}$ .... (ii.),

then $\quad da=0, \quad db+dc=0 \quad \text{and} \quad (s-c)\,db+(s-b)\,dc=0,$

or $\quad b=c, \quad s-b=\tfrac{1}{2}(a+c-b)=\tfrac{1}{2}a,$

and so (ii.) takes the form $sa^2(a-s)+4\Delta^2=0$. It is worth while to note the fact that a and R attain stationary values simultaneously; hence we cannot satisfy $da/d\mathrm{R}=0$. For R is stationary if abc is so, and this gives

$$bc\,da+ca\,db+ab\,dc=0, \quad da+db+dc=0,$$
$$(s-b)(s-c)\,da+(s-c)(s-a)\,db+(s-a)(s-b)\,dc=0;$$

so that
$$\begin{vmatrix} (s-b)(s-c) & bc & 1 \\ (s-c)(s-a) & ca & 1 \\ (s-a)(s-b) & ab & 1 \end{vmatrix} = 0,$$

or $(b-c)(c-a)(a-b)s=0$. If $b=c$, we have the case already discussed. If $a=b$ (say), then $s-c=2a-s$, and so (ii.) gives

$$s(s-2a)(s-a)^2+\Delta^2=0.$$

Let us consider the equation $sx^3-s^2x^2+4\Delta^2=0$. If $x=1/y$,

$$y^3-s^2y/(4\Delta^2)+s/(4\Delta^2)=0.$$

The condition that all the roots should be real is that

$$s^2/(16\Delta^4)-s^6/(432\Delta^6)<0, \quad s^4>27\Delta^2.$$

If this condition is not satisfied, the values of s and Δ are impossible for a real triangle. If it is satisfied, two roots of $sx^2(x-s)+4\Delta^2=0$ are positive, and there is one real maximum and one real minimum, as is geometrically obvious.

15455. (A. M. Nesbitt, M.A.)—Normals OQ, OR are drawn to the ellipse $x^2/a^2+y^2/b^2=1$ from O, the centre of curvature at any point P on the curve. Find the envelope of QR and the locus of its pole.

Solutions (I.) *by* Frances E. Cave *and* R. F. Davis, M.A.; (II.) *by* L. Isserlis, B.A.

(I.) If O be (h, k), the hyperbola $(a^2-b^2)xy-a^2hy+b^2kx=0$ touches the ellipse at P and meets it in Q and R. Therefore, if P be $(a\cos\phi, b\sin\phi)$,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left[(a^2-b^2)xy-a^2hy+b^2kx\right]$$
$$\equiv\left(\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}-1\right)(lx+my+n);$$

therefore the equation of QR is $x/(a\cos\phi)+y/(b\sin\phi)+1=0$. For the envelope $x/a\sec\phi\tan\phi=y/b\operatorname{cosec}\phi\cot\phi$, and hence $\cos\phi=-(x/a)^{\frac{1}{3}}$, $\sin\phi=-(y/b)^{\frac{1}{3}}$; therefore the equation of the envelope is

$$(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1;$$

the pole of QR is $-a/\cos\phi$, $-b/\sin\phi$; and therefore its locus is

$$(a/x)^2+(b/y)^2=1.$$

(II.) Normals at θ, ϕ meet at

$$x=c^2/a\cos\theta\cos\phi\cos\tfrac{1}{2}(\theta+\phi)\sec\tfrac{1}{2}(\theta-\phi),$$
$$y=-c^2/b\sin\theta\sin\phi\sin\tfrac{1}{2}(\theta+\phi)\sec\tfrac{1}{2}(\theta-\phi),$$
$$c^2=a^2-b^2.$$

Put $\theta=\phi$; therefore the centre of curvature at P(θ) is $x=c^2/a\cos^3\theta$, $y=-c^2/b\sin^3\theta$, is O. Let ϕ_1, ϕ_2 be eccentric angles of Q, R. The normal at ϕ is $ax\sec\phi-by\operatorname{cosec}\phi=c^2$, and will pass through O if

$$\cos^3\theta\sin\phi+\sin^3\theta\cos\phi=\sin\phi\cos\phi \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

Put $\tan\frac{1}{2}\phi=t$, $\tan\frac{1}{2}\phi_1=t_1$, $\tan\frac{1}{2}\phi_2=t_2$. (1) can be written

$$\sin^3\theta\, t^4-2(1+\cos^3\theta)t^3+2(1-\cos^3\theta)t-\sin^3\theta=0.$$

Its roots are t_1, t_2, $\tan\frac{1}{2}\theta$, $\tan\frac{1}{2}\theta$; therefore $t_1t_2=-\cot^2\frac{1}{2}\theta=-1/\kappa^2$, say; and $t_1t_2+2(t_1+t_2)\tan\frac{1}{2}\theta+\tan^2\frac{1}{2}\theta=0$; therefore $t_1+t_2=\frac{1}{2}(1/\kappa^3-\kappa)$. The pole of QR is ξ, η where

$$\frac{\xi}{a}=\frac{\cos\frac{1}{2}(\phi_1+\phi_2)}{\cos\frac{1}{2}(\phi_1-\phi_2)}=\frac{1-t_1t_2}{1+t_1t_2}=\frac{\kappa^2+1}{\kappa^2-1},$$

$$\frac{\eta}{b}=\frac{\sin\frac{1}{2}(\phi_1+\phi_2)}{\cos\frac{1}{2}(\phi_1-\phi_2)}=\frac{t_1+t_2}{1+t_1t_2}=-\frac{1}{2}\,\frac{\kappa^4-1}{(\kappa^2-1)\kappa}=-\frac{1}{2}\,\frac{\kappa^2+1}{\kappa};$$

therefore $\kappa=-\xi b/[(\xi-a)\eta]$ and $\kappa^2=(\xi+a)/(\xi-a)$. Hence

$$\eta^2(\xi+a)(\xi-a)=\xi^2b^2$$

(omitting the irrelevant factor $\xi-a$), or

$$a^2/\xi^2+b^2/\eta^2=1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1)$$

is the locus of the pole of OR. The envelope of the polar is the envelope of $x\xi/a^2+y\eta/b^2=1$ with condition (1). I find it to be

$$(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2);$$

it is therefore the evolute of an ellipse whose axes are in the ratio $ab/(a^2-b^2)$ to those of the given ellipse.

The following is the PROPOSER's solution :—

Normals at the ends of the chords $lx+my=1$ and $l'x+m'y=1$ will meet in a point if $a^2ll'=b^2mm'=-1$ (C. Smith's *Conics*, § 198). Now we must take $l'=\cos\phi/a$, $m'=\sin\phi/b$ in order to make the second chord a tangent; whence $al\cos\phi=bm\sin\phi=-1$; so that $1/a^2l^2+1/b^2m^2=1$; and it is under this condition that we have to find the envelope of $lx+my=1$ and the locus of its pole. Taking the latter question first, we have, if (h, k) be the co-ordinates of the pole, $h/a^2=l$, $k/b^2=m$; so that the locus of the pole has for its equation $a^2/h^2+b^2/k^2=1$. In the other differentiate both equations, and we get —

$$x\,dl+y\,dm=0, \quad dl/a^2l^3+dm/b^2m^3=0,$$

and, putting $\lambda=a^2xl^3=b^2ym^3$, we find $\lambda\,(1/a^2l^2+1/b^2m^2)=lx+my=1$; therefore $\lambda=1$; whence the equation to the envelope comes out to be

$$(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1.$$

5637. (Professor W. H. H. HUDSON, M.A.)—A line PQN is drawn, perpendicular to ON, the tangent at O to a curve at a point where the circle of curvature has five-pointic intersection with the curve, cutting the curve and circle of curvature at O in PQ. Prove that PQ varies ultimately as ON^5, and that $PQ/ON^5=(1/120\rho^2)(d^3\rho/ds^3)$ ultimately, where ρ is the radius of curvature at O, and s the arc of the curve measured from a fixed point up to O.

Solution by Professor NANSON.

If κ be the curvature, accents denote differential coefficients with respect to the arc, and $y_r=d^ry/dx^r$, we find by successive differentiation

$$y_1=\tan\psi, \quad y_2=\kappa\sec^3\psi, \quad y_3=\kappa'\sec^4\psi+3\kappa^2\sec^5\psi\sin\psi,$$
$$y_4=\kappa''\sec^5\psi+10\kappa\kappa'\sec^6\psi\sin\psi+3\kappa^3\sec^5\psi\,(5\sec^2\psi-4).$$

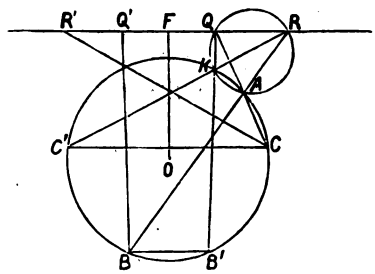
Hence, taking for axes the tangent and normal, we find, by Taylor's theorem, $y=\frac{1}{2}\kappa x^2+\frac{1}{6}\kappa' x^3+\frac{1}{24}(\kappa''+3\kappa^3)\,x^4+\frac{1}{120}(\kappa'''+19\kappa^2\kappa')\,x^5+\dots$.

If now the curve has five-pointic contact with its circle of curvature, we have $\kappa'=0$, $\kappa''=0$, and therefore, if $x=ON$, $PQ=\frac{1}{120}\kappa'''x^5+\dots$. Thus ultimately $PQ\propto ON^5$, and, since κ', κ'' vanish, we have $\rho^2\kappa'''=-\rho'''$, so that ultimately $PQ/ON^5=-\rho'''/120\rho^2$, a result which, apart from sign, is in agreement with the given result.

15730. (JAMES BLAIKIE, M.A.)—ABC is a triangle of which O is the circum-centre, and BC, OA, AB meet a given straight line in P, Q, R F is the foot of the perpendicular from O to the given line;

P′, Q′, R′ are points in the line such that F is the mid-point of PP′, QQ′, RR′. Prove that AP′, BQ′, CR′ meet in a point on the circum-circle of ABC.

Solution by the PROPOSER.



It has been shown in Question 15584 that AP′, BQ′, CR′ meet in a point; it has to be shown that this point is on the circle. Describe a circle about the triangle AQR and let it meet the circle ABC again in K. Join QK, RK, and produce them to meet ABC again in B′, C′. Join BB′, CC′. Then BB′, CC′ are anti-parallel to AK and therefore parallel to QR. If then the points B′, C′, K, Q, R be reflected in the diameter OF, they will take the position B, C, K′, Q′, R′, in which K′ is a point on the circle. Thus BQ′, CR′ (and therefore also AP′) meet on the circle.

15548. (Professor NEUBERG.)—Soient P, Q deux points d'une parabole symétriques par rapport à l'axe de cette courbe. La perpendiculaire élevée en un point quelconque M de la parabole sur la corde PM rencontre le diamètre passant par Q en un point N. Démontrer que la projection de MN sur le diamètre QN est constante.

Solutions (I.) *by* Professor SANJANA, M.A.; (II.) *by* C. M. ROSS.

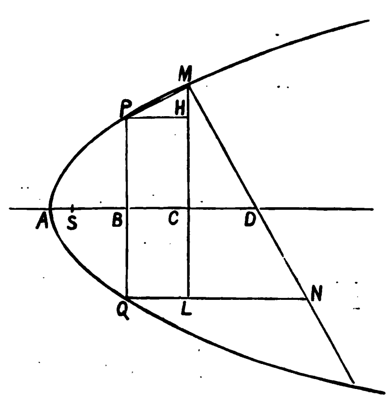
(I.) Making an obvious construction, we have by similar triangles

NL : LM = MH : HP;

hence

$$NL = \frac{LM \cdot MH}{HP} = \frac{MC^2 - HC^2}{HP} = \frac{MC^2 - PB^2}{BC} = \frac{4AS\,(AC - AB)}{BC}$$

= the latus rectum.



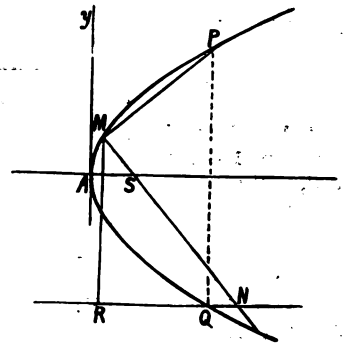
(II.) Let $y^2 = 4ax$ be the equation of the parabola, and $(ak, 2ak)$, $(ak^2, -2ak)$ the points P and Q; also let $(at^2, 2at)$ be the point M. The equation of MN is

$$y - 2at = -\tfrac{1}{2}(k+t)(x-at)^2,$$

and the equation of diameter through Q is $y = -2ak$; then

$$2a(k+t) = \tfrac{1}{2}(k+t)(x-at^2);$$

therefore $x = at^2 + 4a$
$= $ abscissæ of N.



Hence the projection
RN = abscissæ of N
− abscissæ of R
= $4a$ = constant = latus rectum.

15718. (Lt.-Col. Allan Cunningham, R.E.)—Find several cases of numbers (N) expressible by a single digit $(a, b, \ldots)$ repeated not less than three times in two or more scales of radix $(x, y, \ldots)$.
$(N = aa \ldots a = bbb \ldots b = \ldots.)$

Solution by the Proposer.

If there be m repeated digits a, and n repeated digits b, then N is given in two forms $N = a\dfrac{x^m - 1}{x - 1} = b\dfrac{y^n - 1}{y - 1}$ $[a < x,\ b < y;\ m \text{ and } n > 2]$.

Case i.—Taking $m = n = 3$, then the factors of N (except a, b) must consist of factors of form $(A^2 + 3B^2)$. A general solution seems difficult; but particular solutions are readily found by trial, *e.g.*, by aid of the table of factors of $(p^r - 1) \div (p - 1)$ in Euler's *Comment. Arithm.*, Vol. I., pp. 104–109, which give

$$3 . 7 . 13 . 67 = 3 . 7 . \frac{29^3 - 1}{29 - 1} = 13 . \frac{37^3 - 1}{37 - 1};$$

$$3 . 7 . 19 . 67 = 19 . \frac{37^3 - 1}{37 - 1} = \frac{163^3 - 1}{163 - 1};$$

and several others similar.

Case ii.—When one or both of $m, n > 3$, the cases are of course more difficult to find. Two such are

$$31 = \frac{2^5 - 1}{2 - 1} = \frac{5^3 - 1}{5 - 1}; \quad 8 . 3 . 61 = \frac{11^4 - 1}{11 - 1} = 8 . \frac{13^3 - 1}{13 - 1}.$$

10842. (Professor NASH, M.A.)—A variable circle passes through a fixed point A on a conic, and meets the conic again in B, C, D; the Simson-line of A with respect to the triangle BCD passes through a point whose position is independent of the position and magnitude of the circle. This point lies on the diameter through the image of A with respect to the axis; in the rectangular hyperbola it coincides with the centre.

Another Solution by A. M. NESBITT, M.A.

If four points A, B, C, D lie on a circle, the Simson-line of the triangle BCD with respect to the point A is the tangent at the vertex of a parabola inscribed in BCD. Take now A for origin, and for the equation of the fixed conic $S' \equiv ax^2+by^2+2gx+2fy = 0$. Let

$$(px+qy-1)^2 = (p^2+q^2)(x^2+y^2) \quad \text{or} \quad S \equiv (qx-py)^2+2px+2qy-1 = 0$$

be the equation of the parabola passing through BCD. The tangent at its vertex will have for equation $px+qy = \frac{1}{2}$; and between S and S′ we must have the invariant relation $\Theta^2 = 4\Delta\Theta'$. Now

$$\Theta = -(p^2+q^2)(a+b+2qf+2pg), \quad \Theta' = -(fq+gp)^2-ab-2afq-2bgp,$$
$$\Delta = -(p^2+q^2)^2.$$

Hence $(a-b)^2 = 4(a-b)(fq-gp)$; therefore either S′ is a circle or $4(gp-fq)+a-b = 0$. But under this condition the Simson-line of A (whose equation is $2px+2qy = 1$) passes through the fixed point $-2g/(a-b)$, $2f/(a-b)$. Now the axes of co-ordinates, being parallel to the principal axes of S′, are supplemental chords; and thus the diameter through the image of A has for equation $ax/g+by/f+2 = 0$, which clearly passes through the point just determined. If S′ be an equilateral hyperbola, $b = -a$, and the co-ordinates of the fixed point are $-g/a, f/a$, which is obviously the centre.

15716. (A. H. BELL.)—Given $3x+1 = \square$ and $7x+1 = \square$: to find four integral values of x. One of them is 5.

Solutions (I.) *by* R. W. D. CHRISTIE; (II.) *by* G. HEPPEL, M.A.; (III.) *by* ALEXANDER HOLM, M.A.

[It is impossible to publish more than a few of the solutions sent in.—ED.]

(I.) We have $ax+1 = y^2$, $(a+4)x+1 = z^2$, leading to

$$(a+4)y^2-az^2 = 4;$$

thus $\quad y = 1,\ a+1,\ a^2+3a+1,\ \ldots,\ (a+2)q_n-q_{n-1},$

and $\quad z = 1,\ a+3,\ a^2+5a+5,\ \ldots,\ (a+2)q_n-q_{n-1}.$

In the present question $a = 3$; thus

$$y = 1,\ 4,\ 19,\ 91,\ 436,\ \ldots,\ 5q^n-q_{n-1} \text{ ad inf.},$$
$$z = 1,\ 6,\ 29,\ 139,\ 666,\ \ldots,\ 5q_n-q_{n-1} \text{ ad inf.},$$

and consequently $x = 0,\ 5,\ 120,\ 2760,\ 63365,\ 1454640,\ \ldots,$ ad inf.

(II.) $\quad x = \frac{1}{3}(u^2+2u) = \frac{1}{7}(v^2+2v),$

u, v being integers, $\quad v^2+2v+1 = \frac{1}{3}(7u^2+14u+3);$

therefore $\qquad 21u^2+42u+9 = \square = (mu\pm 3)^2$;
therefore, if u_0, u_1 be two integral values of u,

$$u_0 = (42-6m)/(m^2-21), \quad u_1 = (42+6m)/(m^2-21).$$

Eliminating m, $\qquad u_1^2-(5u_0+3)\,u_1+u_0^2-3u_0 = 0.$

Now $u_0 = 0$ gives $x = 0$, and then

$$\begin{array}{ll} u_1^2-3u_1 = 0, & u_1 = 0 \text{ or } 3, \\ u_2^2-18u_2 = 0, & u_2 = 0 \text{ or } 18, \\ u_3^2-93u_3+18\,.\,15 = 0, & u_3 = 3 \text{ or } 90, \\ u_4^2-453u_4+90\,.\,87 = 0, & u_4 = 18 \text{ or } 435, \\ u_5^2-2178u_5+435\,.\,432 = 0, & u_5 = 90 \text{ or } 2088, \\ u_6^2-10443u_6+2088\,.\,2085 = 0, & u_6 = 435 \text{ or } 10008, \end{array}$$

and so on, to any extent.

The values of $x = \frac{1}{2}(u^2+2u)$ are 5, 120, 2760, 63365, 1454640, 33393360. As the inverse process leads back from higher numbers to lower, the list is exhaustive.

(III.) Let $3x+1 = \square = y^2$ and $7x+1 = \square = z^2$. Eliminate x; therefore $\qquad (7y)^2-21z^2 = 28$ ....(1).

(1) When $7y$ is prime to z, $7y = 7$, $z = 1$ is a particular solution, and

$$\sqrt{21} = 4+\frac{1}{\underset{*}{1}+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\underset{*}{8}+},\ \ldots,$$

the penultimate convergent of the cycle being $\frac{55}{12}$. The general primitive integral solution of (1) is

$$7y-z\sqrt{21} = \pm(7-\sqrt{21})(55-12\sqrt{21})^n,$$

where n is zero or any integer positive or negative.

n	y	z	x
0	1	1	0
-1	19	29	120
$+1$	91	169	2760
...	...	...	...

(2) Suppose that $7y$ and z have the common factor 2. Put $7y = 2\mathrm{Y}$ and $z = 2\mathrm{Z}$. Therefore $\qquad \mathrm{Y}^2-21\mathrm{Z}^2 = 7$ ....(2), of which $\mathrm{Y} = 14$, $\mathrm{Z} = 3$ is a particular solution. The general primitive integral solution of (2) is

$$\mathrm{Y}-\mathrm{Z}\sqrt{21} = \pm(14-3\sqrt{21})(55-12\sqrt{21})^n,$$

where n is zero or any integer positive or negative.

n	Y	Z	y	z	x
0	14	3	4	6	5
-1	14	3	4	6	5
$+1$	1526	333	436	666	63365
...	...	...	...	...	...

Not only four integral solutions, but as many as we please, can be obtained. The generalized solutions given of equations (1) and (2) are worthy of note.

10114. (Professor Déprez.)—On considère toutes les coniques inscrites au triangle ABC et dont les axes ont des directions données. (1) Les foyers et les sommets décrivent des cubiques; (2) le lieu d'un point situé sur un axe de l'une des coniques à une distance constante du centre est une conique.

Solution by J. A. H. Johnston, M.A.

This Question is well adapted for solution in oblique co-ordinates. Let the variable conic touch the axes and the line $ax+by+1=0$. Its equation is then

$$(px+qy+1)^2+2\lambda xy=0 \qquad\text{(i.)},$$

and, if $ax+by+1=0$ be a tangent, then $-\lambda = 2(a-p)(b-q)$, and

$$(2pq+\lambda) = 2bp+2aq-2ab \qquad\text{(ii.)}.$$

Let $lx+my=0$ give the assigned direction for a principal axis. Such directions are the bisectors of the line pair $p^2x^2+q^2y^2+2(pq+\lambda)xy=0$, and therefore $lx+my$ is a factor of

$$(pq+\lambda-p^2\cos\omega)\,x^2-(p^2-q^2)\,xy-(pq+\lambda-q^2\cos\omega)\,y^2.$$

It follows that

$$(pq+\lambda-p^2\cos\omega)\,m^2+(p^2-q^2)\,lm-(pq+\lambda-q^2\cos\omega)\,l^2=0 \quad\text{(iii.)}.$$

Equations (i.), with (ii.) and (iii.), completely define the conic and reduce its constants to one arbitrary constant.

The required loci are best investigated by the aid of the locus of the centres. If (x', y') be the centre,

$$\left.\begin{aligned} x' &= \frac{-q}{2pq+\lambda} = \frac{-q}{2(bp+aq-ab)}, \quad y' = \frac{-p}{2pq+\lambda} = \frac{-p}{2(bp+aq-ab)} \\ \text{and therefore}\quad p &= \frac{2aby'}{2ax'+2by'+1}, \quad q = \frac{2abx'}{2ax'+2by'+1} \end{aligned}\right\}\quad\text{(iv.)}.$$

If these values for p and q in (iv.) be substituted in (iii.), the locus of the centres appears as a conic, viz.,

$$2ab\,(l^2\cos\omega-lm)\,x'^2-2ab\,(m^2\cos\omega-lm)\,y'^2+2\,(l^2-m^2)\,abx'y' + (l^2-m^2)(2ax'+2by'+1)=0 \qquad\text{(v.)}.$$

(α) *Locus of a Point on a Principal Axis at a Constant Distance from the Centre.*—Let (x, y) be the point. Then

$$\left.\begin{aligned} l(x-x')+m(y-y') &= 0 \\ \text{and}\quad (x-x')^2+(y-y')^2+2(x-x')(y-y')\cos\omega &= k^2 \end{aligned}\right\}\quad\text{(vi.)}.$$

Equations (vi.) show that both $(x-x')$ and $(y-y')$ have constant values, and, if $x=x'+a'$ and $y=y'+b'$, since (x', y') lies on a conic, it follows that (x, y) also describes a conic.

(β) *Locus of a Focus.*—If x, y be a focus, by well-known equations, (x, y) lies on the conic $4(a-b)\,\phi = \phi_x^2-\phi_y^2$, or, in our case,

$$(p^2-q^2)\left[p^2x^2+q^2y^2+2(pq+\lambda)xy+2px+2qy+1\right] = \left[p^2x+(pq+\lambda)y+p\right]^2-\left[(pq+\lambda)x+q^2y+q\right]^2,$$

and, if this be reduced, it assumes the following simple form:—

$$0 = -(2pq\lambda+\lambda^2)\,x^2+(2pq\lambda+\lambda^2)\,y^2-2q\lambda x+2p\lambda y$$

or

$$x^2-y^2-2x'x+2y'y=0 \qquad\text{(vii.)};$$

therefore (x, y) satisfies

$$l(x-x')+m(y-y')=0 \quad \text{and} \quad x^2-y^2-2x'x+2y'y=0.$$

Solving for x' and y', we find that

$$x'=\frac{m(x^2+y^2)+2lxy}{2(mx+ly)}, \quad y'=\frac{l(x^2+y^2)+2mxy}{2(mx+ly)} \quad \text{.........(viii.).}$$

If these values of x' and y' be now substituted in (v.), we shall find that

$$\begin{aligned}2ab(l^2-m^2)\Big\{lm(x^4+y^4)+2\left[(l^2+m^2)\cos\omega+lm\right]x^2y^2 \\ +(l^2+m^2+2lm\cos\omega)(x^3y+xy^3)\Big\} \\ +2a(l^2-m^2)(mx+ly)(mx^2+my^2+2lxy) \\ +2b(l^2-m^2)(mx+ly)(lx^2+ly^2+2mxy)+2(l^2-m^2)(mx+ly)^2=0.\end{aligned}$$

By testing the terms of the fourth degree, it is easily seen that $mx+ly$ is an irrelevant factor, and the locus appears as

$$\begin{aligned}ab\left[lx^3+my^3+(m+2l\cos\omega)\,x^2y+(l+2m\cos\omega)\,xy^2\right] \\ +(am+bl)(x^2+y^2)+2(al+bm)\,xy+mx+ly=0,\end{aligned}$$

i.e., a cubic which may easily be shown to pass through the vertices of the given triangle of tangents, through the circular points at infinity, and to have a real asymptote parallel to $lx+my=0$.

(γ) *Locus of a Vertex.*—If (x, y) be a vertex, it satisfies

$$l(x-x')+m(y-y')=0,$$

and, moreover, lies on a tangent at right angles to this line. Such a tangent is necessarily

$$(m-l\cos\omega)\,x-(l-m\cos\omega)\,y+\mathrm{K}=0 \quad \text{...............(viii.),}$$

where K must be determined so that (viii.) touches the curve.

It follows from equations (i.) and (ii.) that

$$(a-p)(b-q)=\left(\frac{m-l\cos\omega}{\mathrm{K}}-p\right)\left(\frac{-(l-m\cos\omega)}{\mathrm{K}}-q\right)$$

or that

$$\begin{aligned}(bp+aq-ab)\mathrm{K}^2+\mathrm{K}\left[(l-m\cos\omega)\,p-(m-l\cos\omega)\,q\right]-(l-m\cos\omega)(m-l\cos\omega) \\ =0\text{...(ix.),}\end{aligned}$$

where $\qquad \mathrm{K}=(l-m\cos\omega)\,y-(m-l\cos\omega)\,x.$

Now, by equations (iv.),

$$bp+aq-ab=\frac{-ab}{2ax'+2by'+1},$$

$$(l-m\cos\omega)\,p-(m-l\cos\omega)\,q=\frac{2ab\left[(l-m\cos\omega)\,y'-(m-l\cos\omega)\,x'\right]}{2ax'+2by'+1},$$

and (ix.) becomes

$$\begin{aligned}&-ab\left[(l-m\cos\omega)\,y-(m-l\cos\omega)\,x\right]^2 \\ &+2ab\left[(l-m\cos\omega)\,y-(m-l\cos\omega)\,x\right]\left[(l-m\cos\omega)\,y'-(m-l\cos\omega)\,x'\right] \\ &-(l-m\cos\omega)(m-l\cos\omega)(2ax'+2by'+1)=0 \quad \text{...........................(x.),}\end{aligned}$$

which is linear in (x', y').

Since (x, y) lies on $l(x-x')+m(y-y')=0$, and, moreover,

$$\begin{aligned}2ab(l^2\cos\omega-lm)\,x'^2-2ab(m^2\cos\omega-lm)\,y'^2+2(l^2-m^2)\,abx'y' \\ +(l^2-m^2)(2ax'+2by'+1)=0,\end{aligned}$$

if these two latter equations be solved for x' and y', it will be found that

$$x'=\frac{2ab(m\cos\omega-l)(lx+my)^2-2b(l^2-m^2)(lx+my)-m}{2ab(2lm\cos\omega-l^2-m^2)(lx+my)+2(am-bl)(l^2-m^2)},$$

$$y'=\frac{2ab(l\cos\omega-m)(lx+my)^2+2a(l^2-m^2)(lx+my)+l}{2ab(2lm\cos\omega-l^2-m^2)(lx+my)+2(am-bl)(l^2-m^2)}.$$

When these values for x' and y' are substituted in (x.), since (x.) is linear in x' and y', it easily follows that the locus (x.) is a cubic equation in x and y.

N.B.—To avoid the heavy work entailed in the actual substitutions, it is to be noted that a simple orthogonal projection will place the axes at right angles, and, if the planes intersect in $lx+my=0$, vertices project into vertices, &c. Therefore $\cos\omega$ may be put $=0$, and, if this be done, it will be found that the resulting cubic has one asymptote parallel to $lx+my=0$, and a pair of asymptotes parallel to the perpendicular direction $ly-mx=0$.

15736. (Lt.-Col. Allan Cunningham, R.E.) — (i.) Factorize into *prime* factors $N=(70,600,734^2+1)$. Here $N=q.p^2$, where p is a large prime. (ii.) Show how to find very large numbers ($>10^{50}$) of form

$$N=y^2+1=q.m^2,$$

wherein m is very large ($>10^{25}$). Give examples.

Solution by the Proposer.

The published solutions of the Pellian equation $y^2-Dx^2=-1$ give the minimum integer solution (x, y) for all values of D, which admit of solution in integers, up to $D=1500$. Each of these gives a factorization of $N=y^2+1=D.x^2$. In many of the cases the values of x, y given are very large. Thus Bickmore's table (*Brit. Assoc. Report*, 1893, pp. 73-120) gives

$$D=1213;\quad N=70600734^2+1=1213.2027117^2,$$
$$D=1381;\quad y>2.10^{32},\quad x>5.10^{30},$$
$$D=1453;\quad y>2.10^{38},\quad x>3.10^{36}.$$

Again, taking any such solution of $y^2-Dx^2=-1$, this gives

$$(y^2+Dx^2)^2-D(2xy)^2\equiv+1.$$

Multiplying these together by conformal multiplication gives a new solution $y^2-Dx_1^2=-1$; whence the new number $N_1=y^2+1=Dx_1^2$, where $y_1=y(y^2+3Dx^2)$, $x_1=x(3y^2+Dx^2)$; here the new numbers N_1, y_1, x_1 are much higher than the previous N, x, y. And, by repeated conformal multiplication by the above unit form, the numbers can be raised indefinitely.

15734. (S. C. Gould.)—Give all the different square numbers that can be formed by *all* the ten digits, each taken once and once only. The Proposer has developed eighty-seven such numbers. Are there any more?

Note and Solution by the PROPOSER.

Digital numbers are those that contain the nine digits and each digit but once. There can, therefore, be but 362880 such numbers. But the question arose some sixty or seventy years ago, whether any of these numbers were squares. The problem appeared in several of the newspapers in New England, but to our knowledge no solutions were forthcoming. Finally it appeared in the *Common School Journal* of Connecticut, for January or February, 1858. In the following issue of the *Journal* appeared a solution of the problem by "E. W. R.," Kensington, Conn., giving for the first digital square number $139854276 = 11826^2$; also he explained how he found or developed it. "E. W. R." writes that when he discovered this number he exclaimed, "Eureka!" (I have found it). The problem was stated as follows:—

"Place the nine digits in such a manner that their square root can be extracted without a remainder."

Now it comes to light that, as early as 1727, this "remarkable" number was known; for in an *Arithmetick* now before me (p. 322)* I find the following paragraph:—

"This number 139854276 is a very remarkable number: *firstly*, it is a square number; *secondly*, it contains 9 places, and they are the 9 digits, and I think there is not another that does."

Just thirty of these digital square numbers have been found, and they are given in Table I. following.

Artemas Martin published 27 of the 30 in his *Mathematical Magazine* for January, 1883, and says he copied them from sheets bound in a copy of J. R. Young's *Algebra*, once owned by Abijah McLean, New Lisbon, Ohio, and in his handwriting. Mr. McLean says he obtained his 27 digital squares by the help of Barlow's Table of Square Numbers and the known property that any such square number is divisible by 9, a square number, and consequently the quotient must be a square number. Hence the discovery of the digital squares followed. He also says in connection with his 27 squares: "My investigations have been extended to satisfy me that no more such numbers exist." Yet Mr. McLean missed three of the numbers, namely, the 1st, 26th, and 30th. Dr. James Matteson, a subsequent owner of the *Algebra*, added in his handwriting the 1st digital square, but he does not say that he developed it. Dr. Artemas Martin, Washington, D.C., now owns the book.

In *The Educational Times* of London, in the early seventies, Dr. Martin proposed the question in a new form:—

(No. 3276.) "Give all the different square numbers that can be made with the nine digits, using all the digits once (and only once) in each number."

In 1890, on p. 61 of Vol. LII. of the *Mathematical Questions with their Solutions from The Educational Times* (London), Mr. D. Biddle, Kingston-on-Thames, England, publishes 29 of these numbers; but he did not have the 10th number (382945761).

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* *Arithmetick, both in Theory and Practice; made plain and easie in all the Common and useful rules; &c. The like not extant.* By John Hill, Gent. The Fourth Edition. London, 1727. 8vo; pp. 480.

DIGITAL SQUARES. TABLE I.
(Without the cipher.)

No.	Root. Digital Square.	No.	Root. Digital Square.	No.	Root. Digital Square.
1	11826 = 139854276	11	19629 = 385297641	21	25059 = 627953481
2	12363 = 152843769	12	20316 = 412739856	22	25572 = 653927184
3	12543 = 157326849	13	22887 = 523814769	23	25941 = 672935481
4	14676 = 215384976	14	23019 = 529874361	24	26409 = 697435281
5	15681 = 245893761	15	23178 = 537219684	25	26733 = 714653289
6	15963 = 254817369	16	23439 = 549386721	26	27129 = 735982641
7	18072 = 326597184	17	24237 = 587432169	27	27273 = 743816529
8	19023 = 361874529	18	24276 = 589324176	28	29034 = 842973156
9	19377 = 375468129	19	24441 = 597362481	29	29106 = 847159236
10	19569 = 382945761	20	24807 = 615387249	30	30384 = 923187456

Digital Squares with the Cipher.

In scanning over Table I. of the 30 digital squares some six months ago, it occurred to the editor of the *Notes and Queries* of Manchester, New Hampshire, U.S.A., that for some reasons, suggested by some well known properties of the number 9, if just 30 such squares have been found, and no more, without the 0, then there should be just 90 such digital squares including the cipher.

After reflecting awhile as to some method or process to find them, a quite novel formula presented itself, and after a few trials the following Table II. was easily evolved, but not without some patience, and the making of several thousand figures. But it is remarkable that just 87 digital squares were developed in going through the process once. The editor firmly believes that there are three more such squares to complete 90 such squares, and that he has overlooked them in the process of development. He took the precaution to preserve the manuscript sheets used in finding Table II., and will at leisure review the entire sheets in search of three, more or less, or if any such squares have been overlooked.

In the meantime the last six volumes of the *Notes and Queries* of Manchester, New Hampshire, U.S.A. (1897–1902), will be presented, in numbers, to any person who will discover either one of the three digital squares, if such exist, to make 90 such squares, or possibly more.

Some curious results were noted when Table II. was developed. For instance, the difference of the roots of Nos. 42 and 43 is only 3 ; between the roots of Nos. 47 and 48, 18 ; between the roots of Nos. 27 and 28, 57 ; between the roots of Nos. 75 and 76, 9.

The root of No. 61 (81945) is the reverse of No. 25 (54918).

The root of No. 78 (91248) is twice the root of No. 18 (45624).

In No. 69, the four final figures of the square are the first four digits in order reversed.

In No. 9, the five central figures of the square are the last five digits in order.

In No. 58, the square contains similar arrangements of the digits as in Nos. 69 and 9.

In No. 26 (Table I.), in the square, if the digits 8 and 9 be transposed, and the 0 annexed, the order of the digital figures will be the same as the circulate 7358926410 = $\frac{6690}{9091}$.

DIGITAL SQUARES. TABLE II.

Developed by S. C. Gould, Manchester, N.H.

No.	Root.	Digital Square.	No.	Root.	Digital Square.
1	32043^2	= 1026753849	45	66276^2	= 4392508176
2	32286^2	= 1042385796	46	67677^2	= 4580176329
3	33144^2	= 1098524736	47	68763^2	= 4728350169
4	35172^2	= 1237069584	48	68781^2	= 4730825961
5	35337^2	= 1248703569	49	69513^2	= 4832057169
6	35757^2	= 1278563049	50	71433^2	= 5102673489
7	35853^2	= 1285437609	51	72621^2	= 5273809641
8	37176^2	= 1382054976	52	75759^2	= 5739426081
9	37905^2	= 1436789025	53	76047^2	= 5783146209
10	38772^2	= 1503267984	54	76182^2	= 5803697124
11	39147^2	= 1532487609	55	77246^2	= 5982403716
12	39336^2	= 1547320896	56	78072^2	= 6095237184
13	40545^2	= 1643897025	57	78453^2	= 6154873209
14	42744^2	= 1827049536	58	80361^2	= 6457890321
15	43902^2	= 1927385604	59	80445^2	= 6471398025
16	44016^2	= 1937408256	60	81122^2	= 6597013284
17	45567^2	= 2076351489	61	81945^2	= 6714983025
18	45624^2	= 2081549376	62	83919^2	= 7042398561
19	46587^2	= 2170348569	63	84648^2	= 7165283904
20	48852^2	= 2386517904	64	85353^2	= 7285134609
21	49314^2	= 2431870596	65	85743^2	= 7351862049
22	49353^2	= 2435718609	66	85803^2	= 7362154809
23	50706^2	= 2571098436	67	86073^2	= 7408561329
24	53976^2	= 2913408576	68	86704^2	= 7518029436
25	54918^2	= 3015986724	69	87639^2	= 7680594321
26	55446^2	= 3074258916	70	88623^2	= 7854036129
27	55524^2	= 3082914576	71	89079^2	= 7935068241
28	55581^2	= 3089247561	72	89145^2	= 7946831025
29	55626^2	= 3094251876	73	89355^2	= 7984316025
30	56532^2	= 3195867024	74	89523^2	= 8014367529
31	57321^2	= 3285697041	75	90144^2	= 8125940736
32	58413^2	= 3412078569	76	90153^2	= 8127563409
33	58455^2	= 3416987025	77	90198^2	= 8135679204
34	58554^2	= 3428570916	78	91248^2	= 8326197504
35	59403^2	= 3528716409	79	91605^2	= 8391476025
36	60984^2	= 3719048256	80	92214^2	= 8503421796
37	61575^2	= 3791480625	81	94695^2	= 8967143025
38	61866^2	= 3827401956	82	95154^2	= 9054283716
39	62679^2	= 3928657041	83	96702^2	= 9351276804
40	62961^2	= 3964087521	84	97779^2	= 9560732841
41	63051^2	= 3974528601	85	98055^2	= 9614783025
42	65634^2	= 4307821956	86	98802^2	= 9761835204
43	65637^2	= 4308215769	87	99066^2	= 9814072356
44	66105^2	= 4369871025			

15720. (Professor H. LANGHORNE ORCHARD, M.A., B.Sc.)—**Find the coefficient of n^6 in the product of the two series**

$$1^7+2^7+3^7+4^7+\ldots+(n-1)^7+n^7, \quad 1^8+2^8+3^8+4^8+\ldots+(n-1)^8+n^8.$$

Solution by C. M. Ross.

$$1^7+2^7+3^7+\ldots+n^7 = \tfrac{1}{8}n^8+\tfrac{1}{2}n^7+\text{B}_1\frac{7}{2!}n^6-\text{B}_3\frac{7.6.5}{4!}n^4+\text{B}_5\frac{7.6.5.4.3}{6!}n^2+\text{C}$$
$$= \tfrac{1}{8}n^8+\tfrac{1}{2}n^7+\tfrac{7}{12}n^6-\tfrac{7}{24}n^4+\tfrac{1}{12}n^2,$$

the constant being zero. Also

$$1^8+2^8+3^8+\ldots+n^8 = \tfrac{1}{9}n^9+\tfrac{1}{2}n^8+\text{B}_1\frac{8}{2!}n^7-\text{B}_3\frac{8.7.6}{4!}n^5+\text{B}_5\frac{8.7.6.5.4}{6!}n^3$$
$$-\text{B}_7\frac{8.7.6.5.4.3.2}{8!}n+\text{C}$$
$$= \tfrac{1}{9}n^9+\tfrac{1}{2}n^8+\tfrac{2}{3}n^7-\tfrac{7}{15}n^5+\tfrac{2}{9}n^3-\tfrac{1}{30}n,$$

the constant being zero. Therefore the coefficient of n^6 in the product of the two series is zero.

N.B.—B_1, B_3, B_5, and B_7 are Bernoulli's numbers.

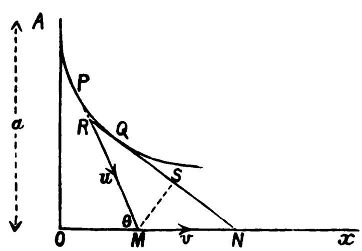
15695. (H. A. Webb, B.A.)—A spider and a fly are a feet apart. The fly starts moving in a direction at right angles to the line joining the animals, and continues moving with uniform velocity v feet per second in a straight line. At the same moment the spider starts moving towards the fly, and continues moving with uniform speed u feet per second ($u>v$) along the "curve of pursuit," *i.e.*, at any moment the spider is moving directly towards the fly. Show that the spider will catch the fly after $au/(u^2-v^2)$ seconds.

Solutions (I.) *by* J. Blaikie, M.A.; (II.) *by* C. Searle; (III.) *by* E. F. Tipple *and* C. Bickerdike.

[Solutions too numerous to allow of nearly all being published.—Ed.]

(I.) Tait and Steele, p. 26, give $x = ae/(1-e^2)$ where $e = v/u$. Thus $x = auv/(u^2-v^2)$. But $v = x/t$; therefore time $= x/v = au/(u^2-v^2)$.

(II.) Let P, Q be the positions of the spider at times t and $(t+\delta t)$ respectively. Draw PM, QN tangents to the "curve of pursuit." M, N are positions of fly at times t and $(t+\delta t)$. Distance of spider from fly = PM = l. Rejecting negligible small quantities, we have



$$\delta l = \text{QN}-\text{PM}$$
$$= (\text{QN}+\text{QR})-(\text{PM}+\text{QR})$$
$$= \text{SN}-(\text{PR}+\text{RQ})$$
$$= \text{SN}-\delta s = v\delta t\cos\theta-\delta s = v\delta t\, dx/ds-\delta s/\delta t\, dt = v/u\, dx-u dt;$$

therefore $l = v/u\, x-ut+\text{C}$. When $t=0$, $x=0$, $l=a$; therefore $l = v/u\, x-ut+a$. When the spider catches the fly $l=0$, $t=\text{T}$, $x=v\text{T}$; therefore $0 = v/u\, v\text{T}-u\text{T}+a$; therefore $\text{T} = ua/(u^2-v^2)$.

(III.) Taking the initial directions of motion as axes, we have

$$u^2 = \left(\frac{dx}{dt}\right)^2 \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}; \quad -\frac{v}{x} = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{dx}{dt}\frac{d^2y}{dx^2},$$

whence path traversed by spider is given by

$$(d^2y/dx^2)/[1+(dy/dx)^2]^{\frac{1}{2}} = v/(ux);$$

therefore, if $z = dy/dx$, $[z+\sqrt{(z^2+1)}]^u a^v = x^v$ or $2z = (x/a)^k - (a/x)^k$, where $k = v/u$ and is < 1; therefore

$$2y = (x/a)^k x/(k+1) - (a/x)^k x/(1-k) - a/(k+1) + a/(1-k).$$

Thus, when $x = 0$, $y = ak/(1-k^2)$, and time taken by fly to arrive at this point $= au/(u^2-v^2)$. Time taken by spider to reach same point

$$= \int \frac{ds}{u} = \frac{1}{2u}\left\{\left(\frac{x}{a}\right)^k \frac{x}{1+k} + \left(\frac{a}{x}\right)^k \frac{x}{1-k}\right\}_0^a = \frac{au}{u^2-v^2}.$$

15075. (H. L. Trachtenberg.)—The straight line joining the centres of the two rectangular hyperbolas that touch four fixed straight lines is bisected at right angles by the directrix of the parabola which touches these straight lines.

Solutions (I.) *by the* Proposer, (II.) *by* Professor Sanjána, M.A., (III.) *by* W. H. Blythe, M.A.

(I.) The director circles of all the conics touching the four fixed straight lines form a coaxal system. But the centres of the two rectangular hyperbolas are their director circles and are therefore the limiting points of the system. Also the directrix of the parabola is its director circle and is therefore the radical axis of the system, and, since the join of the limiting points is bisected at right angles by the radical axis, it follows that the join of the centres of the two rectangular hyperbolas is bisected at right angles by the directrix of the parabola.

(II.) Of the four triangles formed by the fixed straight lines taken three and three, two must be obtuse-angled. Draw the two circles for which these two triangles are severally self-conjugate: then either point of intersection of the circles is the centre of a rectangular hyperbola touching the given lines. Thus the line of centres of the hyperbolas is bisected at right angles by the line of centres of the circles, *i.e.*, by the line joining the orthocentres of the two triangles. But, since the parabola touches the four given lines, the orthocentres of the four triangles lie all on its directrix; hence this directrix bisects at right angles the line of centres of the hyperbolas.

(III.) The following may be interesting, as it gives the analytical solution of the problem. A geometrical proof based on Cremona, *Projective Geometry*, 2nd ed., pp. 271 *et seq.*, is intrinsically superior, however. It is shown in Salmon's *Conic Sections*, p. 254, that the conic

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 0$$

touches the four straight lines $l\alpha \pm m\beta \pm n\gamma = 0$ if $l^2/a + m^2/b + n^2/c = 0$, and that the locus of the centres of the conics is $l^2\alpha + m^2\beta + n^2\gamma = 0$. Taking areal co-ordinates and expressing the condition that the conic may be a rectangular hyperbola, we have $a + b + c = 0$. If x, y, z are the co-ordinates of the centre, then $ax = by = cz$, for it is the pole of the line at infinity. Using these equations with those given above to find the co-ordinates of the centre, we find, in addition to the fact that x, y, z lies on the line of centres, that $xy + yz + zx = 0$. Therefore there are but two rectangular hyperbolas the centres of which lie at the intersection of the line of centres with the circle circumscribing the triangle of reference. Taking the condition that the conic should be a parabola, namely, $ab + bc + ca = 0$, the two tangents drawn from the centre of the circumscribing circle $\alpha = \beta = \gamma$ and given by the equation

$$(a\alpha^2 + b\beta^2 + c\gamma^2)(a + b + c) = (a\alpha + b\beta + c\gamma)^2$$

prove to be at right angles when the above condition is satisfied. Therefore the directrix of the parabola passes through the centre of the circle, and, being evidently at right angles to the line of centres, it bisects the chord of the circle made by this line, that is, the line joining the centres of the rectangular hyperbolas.

12952. (W. Booth.)—If Q stands for

$$ax^2 + 2hxy + by^2 + 2gzx + 2fyz + cz^2 + 2lx\omega + 2my\omega + 2nz\omega + d\omega^2,$$

then the determinant $\begin{vmatrix} aQ - L^2, & hQ - LM, & gQ - LN \\ hQ - ML, & bQ - M^2, & fQ - MN \\ gQ - LN, & fQ - MN, & cQ - N^2 \end{vmatrix}$ is equal to $\Delta Q^2 \omega^2$, where $L = \frac{1}{2}(dQ/dx)$, $M = \frac{1}{2}(dQ/dy)$, $N = \frac{1}{2}(dQ/dz)$. Give a geometrical interpretation.

Solution by Professor Nanson.

The condition that the tangent cone from the point x to the conicoid

$$u \equiv \Sigma a_{pq} x_p x_q = 0 \quad (p, q = 1, 2, 3, 4)$$

may intersect the plane $\quad \alpha \equiv \Sigma a_p x_p = 0$

in a line pair is, if $\quad u_p = \frac{1}{2}(du/dx_p)$,

$$\begin{vmatrix} (ua_{pq} - u_p u_q) & a_p \\ (a_q) & 0 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} (ua_{pq} - u_p u_q) & (a_p) & (0) \\ (a_q) & 0 & 0 \\ (u_q) & 0 & 1 \end{vmatrix} = 0,$$

and by simple combinations of rows and columns this reduces to

$$1/u^2 \begin{vmatrix} (ua_{pq}) & (a_p) & (0) \\ (a_q) & 0 & -\alpha \\ (0) & -\alpha & 0 \end{vmatrix} = 0 \quad \text{or} \quad \Delta u^2 \alpha^2 = 0,$$

i.e., either the conicoid is a cone or the vertex of the tangent cone lies on u or on α. By taking $\alpha \equiv x_4$, we get the result in the Question.

[N.B.—The notation of the Question has not been entirely followed.]

15741. (R. CHARTRES.)—From a point within a triangle straight lines parallel to the sides are drawn to the base. Find the mean value of the n-th power of the area of the triangle thus formed. (Elementary proof wanted.)

Solution by the PROPOSER.

$$a^{2n} = (p+q+r)^{2n},$$

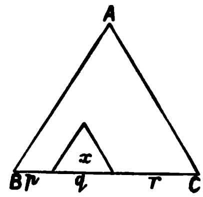
having $(n+1)(2n+1)$ terms; therefore

$$M(q^{2n}) = \frac{a^{2n}}{(n+1)(2n+1)},$$

or

$$M(x^n) = \frac{\Delta^n}{(n+1)(2n+1)}.$$

If $n = 1$, then $M(x) = \frac{1}{6}\Delta$.



15704. (R. F. WHITEHEAD, B.A.)—Expand $\theta/\sin\theta$ in ascending powers of $\cos\theta$.

Solutions (I.) *by* Professor SANJÁNA, M.A.; (II.) *by* R. F. DAVIS, M.A.

It is well known that $\dfrac{\sin^{-1}x}{\sqrt{(1-x^2)}} = x + \frac{2}{3}x^3 + \dfrac{2.4}{3.5}x^5 + \ldots$.

Putting $x = \cos\theta$, $\dfrac{\frac{1}{2}\pi-\theta}{\sin\theta} = \cos\theta + \frac{2}{3}\cos^3\theta + \dfrac{2.4}{3.5}\cos^5\theta \ldots$; also

$$\frac{\frac{1}{2}\pi}{\sin\theta} = \tfrac{1}{2}\pi(1-x^2)^{-\frac{1}{2}} = \tfrac{1}{2}\pi\left(1 + \tfrac{1}{2}\cos^2\theta + \frac{1.3}{2.4}\cos^4\theta \ldots\right).$$

Hence, by subtraction,

$$\frac{\theta}{\sin\theta} = \tfrac{1}{2}\pi - \cos\theta + \tfrac{1}{2}\pi\,\tfrac{1}{2}\cos^2\theta - \tfrac{2}{3}\cos^3\theta + \tfrac{1}{2}\pi\frac{1.3}{2.4}\cos^4\theta - \frac{2.4}{3.5}\cos^5\theta \ldots.$$

(II.) Assume $\theta/\sin\theta = A_0 + A_1\cos\theta + A_2\cos^2\theta + \ldots$. Then, if θ and ϕ be complementary, $\frac{1}{2}\pi - \phi = \cos\phi(A_0 + A_1\sin\phi + A_2\sin^2\phi + \ldots)$ and $A_0 = \frac{1}{2}\pi$. Differentiating,

$$-1 = -\sin\phi(A_0 + A_1\sin\phi + A_2\sin^2\phi + \ldots) + (1-\sin^2\phi)(A_1 + 2A_2\sin\phi + 3A_3\sin^2\phi + \ldots).$$

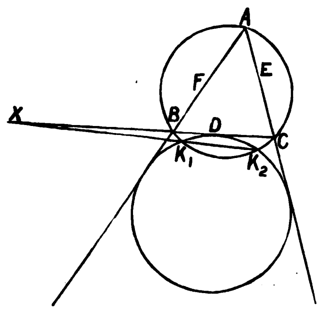
Equating coefficient of $\sin^{n-1}\phi$ to zero, $-A_{n-2} + nA_n - (n-2)A_{n-2} = 0$, or $nA_n = (n-1)A_{n-2}$. Thus

$$\theta/\sin\theta = \tfrac{1}{2}\pi\left(1 + \tfrac{1}{2}\cos^2\theta + \frac{1.3}{2.4}\cos^4\theta + \ldots\right) - \left(\cos\theta + \tfrac{2}{3}\cos^3\theta + \frac{2.4}{3.5}\cos^5\theta + \ldots\right).$$

2361. (Rev. R. TOWNSEND, F.R.S.)—(1) Show that the three chords of intersection of the circumscribed with the three escribed circles of a plane triangle intersect collinearly with the three corresponding sides

of the triangle. (2) Prove the corresponding property for a spherical triangle.

Solution by Professor SANJÁNA, M.A.



(1) Let the common chord with the A-escribed circle meet BC in X, and let D be the point of contact in that side. Then, as X is on the radical axis,

$$XB \, . \, XC = XD^2;$$

therefore

$$XC/XD = XD/XB = (XC-XD)/(XD-XB) = CD/DB,$$

and hence

$$CD^2/DB^2 = XC/XB.$$

Similar results hold for the other sides. Therefore

$$\frac{XC}{XB} \cdot \frac{YA}{YC} \cdot \frac{ZB}{ZA} = \frac{CD^2}{DB^2} \cdot \frac{AE^2}{EC^2} \cdot \frac{BF^2}{FA^2} = \frac{(s-b)^2}{(s-c)^2} \cdot \frac{(s-c)^2}{(s-a)^2} \cdot \frac{(s-a)^2}{(s-b)^2} = 1;$$

so that X, Y, Z are collinear.

(2) If the triangle is spherical, X is the intersection with BC of the great circle through K_1 and K_2, and $\tan^2 \frac{1}{2}XD = \tan \frac{1}{2}XB \tan \frac{1}{2}XC$; therefore

$$\frac{\tan \frac{1}{2}XC}{\tan \frac{1}{2}XD} = \frac{\tan \frac{1}{2}XD}{\tan \frac{1}{2}XB} = \frac{\tan \frac{1}{2}XC - \tan \frac{1}{2}XD}{\tan \frac{1}{2}XD - \tan \frac{1}{2}XB} = \frac{\sin \frac{1}{2}CD \cos \frac{1}{2}XB}{\sin \frac{1}{2}BD \cos \frac{1}{2}XC},$$

and hence $$\frac{\sin^2 \frac{1}{2}CD \cos^2 \frac{1}{2}XB}{\sin^2 \frac{1}{2}BD \cos^2 \frac{1}{2}XC} = \frac{\sin \frac{1}{2}XC \cos \frac{1}{2}XB}{\sin \frac{1}{2}XB \cos \frac{1}{2}XC},$$

i.e., $$\frac{\sin^2 \frac{1}{2}CD}{\sin^2 \frac{1}{2}BD} = \frac{\sin XC}{\sin XB};$$

and similar results hold for the other sides. Therefore

$$\frac{\sin XC}{\sin XB} \cdot \frac{\sin YA}{\sin YC} \cdot \frac{\sin ZB}{\sin ZA} = \frac{\sin^2 \frac{1}{2}(s-b)}{\sin^2 \frac{1}{2}(s-c)} \cdot \frac{\sin^2 \frac{1}{2}(s-c)}{\sin^2 \frac{1}{2}(s-a)} \cdot \frac{\sin^2 \frac{1}{2}(s-a)}{\sin^2 \frac{1}{2}(s-b)} = 1;$$

so that X, Y, Z lie on a great circle.

15824. (Professor H. LANGHORNE ORCHARD, M.A., B.Sc.)—Prove that

$$11(1 + 2^{10} + 3^{10} + 4^{10} + \dots + n^{10}) - 10n(1 + 2^9 + 3^9 + 4^9 + \dots + n^9)$$
$$= \tfrac{1}{6}(3n^{10} + 10n^9 - 24n^7 + 36n^5 - 24n^3 + 5n).$$

Solution by CONSTANCE I. MARKS, B.A.

We have, by the usual method for finding the sums of powers of the first n natural numbers,

$$\Sigma n^r = [n^{r+1} + {}^{r+1}C_2 \Sigma n^{r-1} - {}^{r+1}C_3 \Sigma n^{r-2} + \dots + (-1)^{s+1}\, {}^{r+1}C_{s+1} \Sigma n^{r-s} + \dots$$
$$+ (-1)^{r+1} n]/(r+1)$$

where r and s are integers, $s \not> r$. If we give to r the values 1, 2, 3, ..., 10, we get successively

$$\Sigma n = \tfrac{1}{2}(n^2+n); \quad \Sigma n^2 = \tfrac{1}{3}(n^3+\tfrac{3}{2}n^2+\tfrac{1}{2}n); \quad \Sigma n^3 = \tfrac{1}{4}(n^4+2n^3+n^2);$$
$$\Sigma n^4 = \tfrac{1}{5}(n^5+\tfrac{5}{2}n^4+\tfrac{5}{3}n^3-\tfrac{1}{6}n); \qquad \Sigma n^5 = \tfrac{1}{6}(n^6+3n^5+\tfrac{5}{2}n^4-\tfrac{1}{2}n^2);$$
$$\Sigma n^6 = \tfrac{1}{7}(n^7+\tfrac{7}{2}n^6+\tfrac{7}{2}n^5-\tfrac{7}{6}n^3+\tfrac{1}{6}n); \quad \Sigma n^7 = \tfrac{1}{8}(n^8+4n^7+\tfrac{14}{3}n^6-\tfrac{7}{3}n^4+\tfrac{2}{3}n^2);$$
$$\Sigma n^8 = \tfrac{1}{9}(n^9+\tfrac{9}{2}n^8+6n^7-\tfrac{21}{5}n^5+2n^3-\tfrac{3}{10}n).$$
$$10n\Sigma n^9 = n^{11}+5n^{10}+\tfrac{15}{2}n^9-7n^7+5n^5-\tfrac{3}{2}n^3;$$
$$11\Sigma n^{10} = n^{11}+\tfrac{11}{2}n^{10}+\tfrac{55}{6}n^9-11n^7+11n^5-\tfrac{11}{2}n^3+\tfrac{5}{6}n;$$

therefore
$$11\Sigma n^{10}-10n\Sigma n^9 = \tfrac{1}{2}n^{10}+\tfrac{10}{6}n^9-4n^7+6n^5-4n^3+\tfrac{5}{6}n$$
$$= \tfrac{1}{6}(3n^{10}+10n^9-24n^7+36n^5-24n^3+5n).$$

15714. (Robert W. D. Christie.) — Multiply 567×543 in three operations and prove the general theory. (Either number at top.)

Ex.—

$$\begin{array}{r} 56\ 7 \\ \underline{54\ 3} \\ 3078{\cdot}21 \\ \underline{6} \\ \underline{\underline{3078{\cdot}81}} \end{array} \qquad \begin{array}{r} 54\ 3 \\ \underline{56\ 7} \\ 3080{\cdot}21 \\ \underline{-1\ 4} \\ \underline{\underline{3078{\cdot}81}} \end{array}$$

Solution by the Proposer.

By ordinary multiplication $(10x+y)(10v+z) = 10^2vx+10vy+10xz+yz$. Now $10vy = 10v(10-z) = 10^2v-10vz$ (since $y+z = 10$); therefore

$$(10x+y)(10v+z) = 10^2v(x+1)+10z(x-v)+yz,$$

which is the theorem.

The following gives the extension to all numbers:—

$$(10x+y)(10v+z) = 10^2v(x+1)+10z(x-v)+(y+z-10)\,10v+yz.$$

(1) Here note if $x = v$ or $y+z = 10$ the operations are unnecessary; *e.g.*,
$$\begin{array}{r} 8{\cdot}7 \\ \underline{8{\cdot}3} \\ \underline{\underline{72{\cdot}21}} \end{array}$$
or $8 \times 9 = 72$ and $3 \times 7 = 21$, leaving two operations only.

(2) The principle applies to 10^n as well as to 10^1, thus

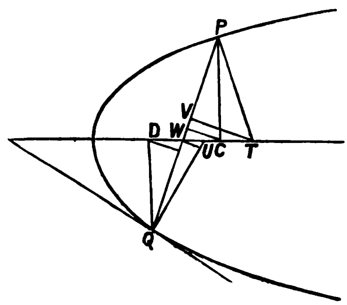
$$\begin{array}{r} 8{\cdot}76 \\ \underline{8{\cdot}24} \\ \underline{\underline{72{\cdot}1824}} \end{array} \qquad \text{here } 76+24 = 10^2.$$

(3) Reversing the principle, we may factorize $9702{\cdot}044791$. Here the product of two consecutives 98×99, or $x^2+x = 9702$ and $ab = {\cdot}044791$, where $a+b = 10^3$, whence a and b. The factors are 98953 and 98047.

15726. (Professor Sanjána, M.A.)—Show that the normals drawn at the extremities of any chord of a parabola and terminated by the axis

have equal projections upon that chord; and that these projections are constant when the chord moves so that the algebraical difference of the ordinates of the two extremities has a constant projection on the chord. Prove also that this condition is satisfied by every focal chord.

Solution by C. M. Ross.



Let the equation of the parabola be $y^2 = 4ax$, and let $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ be any two points on it. The equations of the normals at these points are

$$t_1x + y = at_1(2 + t_1^2) \ \ldots(1),$$

$$t_2x + y = at_2(2 + t_2^2) \ \ldots(2).$$

(1) intersects the x-axis in the point $[a(2 + t_1^2), 0] \ldots(\alpha)$,

(2) intersects the x-axis in the point $[a(2 + t_2^2), 0] \ldots(\beta)$.

Again the equation of the chord PQ is

$$2x - y(t_1 + t_2) + 2at_1t_2 = 0;$$

then the distance of (α) from PQ is $2a \dfrac{2 + t_1t_2 + t_1^2}{\sqrt{[4 + (t_1 + t_2)^2]}}$; also the distance of (β) from PQ is $2a \dfrac{2 + t_1t_2 + t_2^2}{\sqrt{[4 + (t_1 + t_2)^2]}}$.

Again $PT = 2a\sqrt{(1 + t_1^2)}$ and $QV = 2a\sqrt{(1 + t_2^2)}$; therefore

$$PV^2 = PT^2 - TV^2 = 4a^2(t_1 - t_2)^2/[4 + (t_1 + t_2)^2].$$

Similarly $QW^2 = 4a^2(t_1 - t_2)^2/[4 + (t_1 + t_2)^2]$; therefore the projections of T and U on the chord are equal.

Again the distance of C (the foot of the ordinate of P) from PQ is $2at_1(t_1 + t_2)/\sqrt{[4 + (t_1 + t_2)^2]}$; also the distance of D from PQ is

$$2at_2(t_1 + t_2)/\sqrt{[4 + (t_1 + t_2)^2]};$$

therefore the projections of C and D on PQ are

$$4at_1/\sqrt{[4 + (t_1 + t_2)^2]} \quad \text{and} \quad 4at_2/\sqrt{[4 + (t_1 + t_2)^2]}.$$

Now the algebraical difference of these projections is

$$4a(t_1 - t_2)/\sqrt{[4 + (t_1 + t_2)^2]} = K \ \text{(constant)};$$

therefore $PV = QW = \frac{1}{2}K$, which is constant also.

Every focal chord satisfies the above condition, since in this case $t_2 = -1/t_1$.

15656. (James Blaikie, M.A.)—If a straight line drawn through the circum-centre of a triangle ABC meet BC, CA, AB in P, Q, R, and if points P′, Q′, R′ be taken on the line so that O is the mid-point of

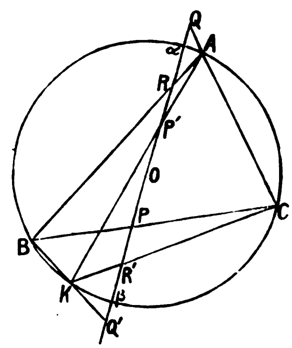
PP′, QQ′, RR′, prove that AP′, BQ′, CR′ meet at a point in the circum-circle.

Solutions (I.) *by* Professor Sanjána, M.A.; (II.) *by* W. F. Beard, M.A.

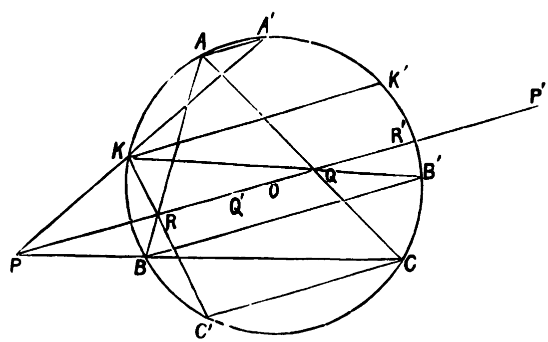
(I.) Let PQR cut the circle in α and β. With regard to the two points α and β, P and P′, Q and Q′, R and R′ are isotomic; therefore, by a theorem previously given by me, AP′, BQ′, CR′ meet in a point: let this be K. Since

$$(PQ \,.\, PQ')/(P'Q \,.\, P'Q') = 1$$
$$= (P\alpha \,.\, P\beta)/(P'\alpha \,.\, P'\beta),$$

therefore (PP′, QQ′, $\alpha\beta$) is an involution; so also (QQ′, $\alpha\beta$, RR′) is an involution. Hence PP′QQ′RR′$\alpha\beta$ is a range in involution; that is to say, the opposite connectors of the quadrangle ABKC and the circle ABC determine a range in involution on the line PQR; hence the quadrangle must be cyclic. Therefore K lies on the circum-circle of ABC.



(II.) As PRQ is a transversal, it is easily proved that the circum-circles of the triangles AQR, BRP, CPQ, ABC meet at a point. Let

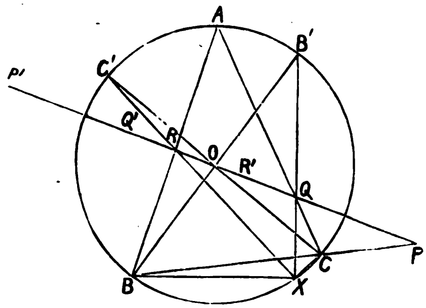


this point be K. Join PK, QK, RK and produce them to meet the circumference at A′B′C′. Because A, Q, R, K are on the circle, therefore $\angle$KQR = $\angle$KAR = $\angle$KB′B (Euc. III. 21); therefore BB′ is parallel to PQ. Similarly AA′, CC′ are parallel to PQ. Draw KK′ parallel to PQ. Then, since AA′, KK′ are parallel to PQ and OP′ = OP, it is plain by symmetry that, as A′K passes through P, therefore AK′ passes through P′. Thus AP′, BQ′, CR′ all pass through K′, a point on the circum-circle of ABC.

[The Proposer solves the problem as follows:—

As shown in Problem 15633, *vide Reprint*, Vol. VIII. (New Series), p. 40, and *Educational Times*, February, 1905, if BB′ and CC′ be

diameters, then, by the converse of Pascal's theorem, since QOR is a straight line, B′Q and C′R meet at a point in the circle. Call this point



X. If now the whole figure be rotated through 180° about O, the points B′, C′, C, R will come into the positions B, C, Q′, R′. Thus BQ′, CR′ (and similarly AP′) meet in the point on the circum-circle diametrically opposite to X.]

15738. (Professor Nanson.) — Eliminate λ from $\Sigma[a_r/(c_r+\lambda)] = 0$, $\Sigma[b_r/(c_r+\lambda)] = 0$, where $r = 1, 2, \ldots, n$.

Solution by J. A. H. Johnston, M.A., *and* C. M. Ross.

Let $\quad 1/(c_1+\lambda) = X_1, \; \ldots, \; 1/(c_n+\lambda) = X_n, \; \ldots.$

Then, by the conditions, we have the following $(n+1)$ equations:—

$$\left.\begin{array}{ll} a_1X_1 + a_2X_2 + \ldots a_nX_n = 0, & \\ b_1X_1 + b_2X_2 + \ldots b_nX_n = 0, & \\ X_1 - \; X_2 & + (c_1-c_2)X_1X_2 = 0, \\ X_1 + \ldots 0 - X_3 & + (c_1-c_3)X_1X_3 = 0, \\ \vdots & \\ X_1 & -X_n + (c_1-c_n)X_1X_n = 0. \end{array}\right\} \ldots\ldots \text{(i.)}$$

Let us regard the products $X_1X_2, \ldots, X_1X_n \ldots$ as constants and eliminate $X_1, X_2, \ldots, X_n$. We therefore get

$$\begin{vmatrix} a_1 & a_2 & & a_n & 0 \\ b_1 & b_2 & & b_n & 0 \\ 1 & -1 & & 0 & (c_1-c_2)X_1X_2 \\ 1 & 0 & -1 & 0 & (c_1-c_2)X_1X_3 \\ \vdots & & & \vdots & \ldots \\ 1 & 0 & 0 & -1 & (c_1-c_n)X_1X_n \end{vmatrix} = 0, \qquad \text{(ii.)}$$

If the common factor X_1 be rejected, this determinant is a linear equation in $(X_2, X_3, \ldots, X_n)$.

If equations (i.) be now rewritten, so that the last $(n-1)$ equations contain X_2 and X_1, X_2 and X_3, ..., X_2 and X_n, and the variables again eliminated, the result is another determinant from which X_2 may be rejected, giving another linear equation in $X_1, X_3, ..., X_n$. This process may be continued until we have formed $(n-2)$ independent equations in each of which one variable is absent, and, since, from the form of (ii.), these equations are all homogeneous, if they be combined with the first pair of equations (i.), the eliminant appears as the determinant given by the n homogeneous equations thus formed. An equivalent result appears by eliminating $X_1 \dots X_n$ between the n equations of the form of (ii.).

15510. (Rev. T. Roach, M.A.)—Give a proof of the known equality $\tanh^{-1}[(i \sin\theta)/i] = \log\tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$, and hence show that

$$i\tan^{-1} i = -\infty.$$

Solution by W. Scrimgeour, M.A., B.Sc.

$$\tanh^{-1}\left(\frac{i\sin\theta}{i}\right) = \tanh^{-1}\left(\frac{i[-\sin(-\theta)]}{i}\right)$$

$$= \tanh^{-1}\left(\frac{i[-\sin i(i\theta)]}{i}\right)$$

$$= \tanh^{-1}\left(\frac{i[-i\sinh(i\theta)]}{i}\right) \quad (\text{for } \sin ix = i\sinh x)$$

$$= \tanh^{-1}\left(\frac{-i^2\sinh(i\theta)}{i}\right)$$

$$= \tanh^{-1}\left(\frac{\sinh(i\theta)}{i}\right)$$

$$= \tanh^{-1}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)$$

$$= \tanh^{-1}(\sin\theta)$$

$$= \tfrac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta} = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta).$$

In the second part of the question $i\tan^{-1} i = -\infty$; for let $\tan x = i$;

therefore $\quad x = \tan^{-1} i; \quad \dfrac{e^{ix}-e^{-ix}}{i(e^{ix}+e^{-ix})} = i; \quad \dfrac{e^{2ix}-1}{e^{2ix}+1} = -1;$

therefore $e^{2ix}-1 = -e^{2ix}-1$; therefore $2e^{2ix} = 0$; therefore $e^{ix} = 0$; therefore $ix = -\infty$; therefore $i\tan^{-1} i = -\infty$.

15752. (Professor M. W. Crofton, F.R.S.)—Show that $n^n - 1$ is divisible by $4n+1$ if the latter is a prime number.

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E., (II.) *by* R. W. D. Christie; (III.) *by* F. N. Mayers, M.A.

(I.) This is a known theorem. It may be proved thus, $2^2 . n \equiv -1$, and $2^4 . n \equiv -2^2 \pmod p$; therefore

$$(2^4)^{\frac14(p-1)} n^{\frac14(p-1)} \equiv (-1)^{\frac14(p-1)} (2^2)^{\frac14(p-1)} \quad (\text{mod } p).$$

Herein $2^{p-1} \equiv +1$, $\frac14(p-1) = n$; and $(-1)^{\frac14(p-1)}$, $2^{\frac12(p-1)}$ both $\equiv -1$ when n is odd, and both $\equiv +1$ when n is even; therefore $n^n \equiv 1 \pmod p$ always.

(II.) (1) Let n be odd. We have

$$4n \equiv -1 \text{ (mod) } 4n+1,$$

i.e., $$4^n n^n = -1,$$

$$4^n n^{2n} = -n^n = -1 \text{ (hyp.)}.$$

$$4^n . n^{2n} - 4n \equiv 0 \text{ (mod) } 4n+1;$$

i.e., $$4^{n-1} n^{2n-1} \equiv +1 \text{ (mod) } 4n+1 \text{ (even)},$$

$$4^{n-2} n^{2n-2} \equiv -1 \text{ (mod } 4n+1) \text{ (odd)},$$

$$4' n^{n+1} \equiv -1 \text{ (mod } 4n+1) \text{ odd};$$

$$4^0 n^n \equiv +1$$

or $$n^n \equiv +1 \text{ as per hypothesis.}$$

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(2) Let n be even $$4n \equiv -1 \text{ (mod } p),$$

$$4^n n^n \equiv +1 \text{ (mod } p),$$

$$4^n n^{2n} \equiv n^n \equiv +1 \text{ (hyp.)},$$

$$4^{n-1} n^{2n-1} \equiv -1 \text{ (odd)},$$

$$4^{n-2} n^{2n-2} \equiv +1 \text{ (even)},$$

$$4' n^{n+1} \equiv -1 \text{ odd},$$

$$n^n = +1, \text{ mod } 4n+1, \text{ as before.}$$

I may here give an interesting deduction from the working. By Wilson's theorem we have $(4n)! \equiv -1$, mod $4n+1$, and $4n \equiv -1$, thus

$$(4n-1)! = 1 \quad (\text{mod } 4n+1),$$

and the above theorem leads to others.

(III.) It is evident that the problem amounts to showing that n is a biquadratic residue of $4n+1$; for, if $n \equiv a^4$, $n^n \equiv a^{4n} \equiv 1 \,(\text{mod } 4n+1)$; or if we denote $4n+1$ by p, that $\frac14(p-1)$ is a biquadratic residue of p. If we divide the residues of p into four classes A, B, C, D, being respectively of the form g^{4n}, g^{4n+1}, g^{4n+2}, g^{4n+3}, g being a primitive root of p, as is done by Gauss in his *Theoria Residuorum biquadraticorum*, it is evident that class A are all biquadratic residues, class C quadratic residues and biquadratic non-residues, and the other two classes are non-quadratic residues. It is evident that the quotient of any two numbers of the same classes one by the other will be a biquadratic residue, for it will be of the form g^{4l}.

(1) If p is of the form $8n+5$, 2 is a non-residue, and 4 therefore belongs to class C. But -1, from the well known biquadratic character of -1, belongs to the same class. For $(-1)^{\frac{1}{4}(p-1)} = -1$; $\frac{1}{4}(p-1)$ being odd. Hence $\frac{1}{4}(p-1)$ belongs to class A.

(2) If p is of the form $8n+1$, both $p-1$, $(-1)^{\frac{1}{4}(p-1)} = +1$, $\frac{1}{4}(p-1)$ being even, and 4 belong to class A; for in this case 2 is a quadratic residue; so that in this case $\frac{1}{4}(p-1)$ obviously belongs to class A.

The classes A, B, C, D are defined by Gauss as being such that, if α belongs to the first, second, third, or fourth class, $(\alpha)^{\frac{1}{4}(p-1)}$ will be congruent to $1, f, -1, -f$ respectively; these four quantities being the four roots of $a^4 \equiv 1 \pmod p$.

15657. (Professor Neuberg.)—Les diagonales AC, BD d'un quadrilatère ABCD se coupent à angle droit en O. On sait que les projections de O sur les côtés du quadrilatère sont situées sur une même circonférence. Si p est la puissance de O par rapport à cette circonférence et R le rayon, démontrer que

$$1/p = 1/(\mathrm{OA}\,.\,\mathrm{OC}) + 1/(\mathrm{OB}\,.\,\mathrm{OD}),$$
$$4\mathrm{R}^2/p^2 = (1/\mathrm{OA}-1/\mathrm{OC})^2 + (1/\mathrm{OB}-1/\mathrm{OD})^2.$$

Solutions (I.) *by the* Proposer; (II.) *by* C. M. Ross.

(I.) Soient $(a, 0)$, $(0, b)$, $(a', 0)$, $(0, b')$ les co-ordonnées des points A, B, C, D. Les équations de la droite AB et la perpendiculaire $O\alpha$ étant $x/a+y/b = 1$, $y = (a/b)\,x$, les coordonnées de α sont $x = ab^2/(a^2+b^2)$, $y = a^2b/(a^2+b^2)$. Exprimons que ce point est sur la circonférence

$$x^2+y^2-2mx-2ny+p = 0\,;$$

on trouve $\qquad 1-2m/a-2n/b+p\,(1/a^2+1/b^2) = 0$ ....(1).

Par analogie, si β, γ, δ sont sur la même circonférence, on a

$$1-2m/a'-2n/b+p\,(1/a'^2+1/b^2) = 0 \quad\text{..................(2)},$$
$$1-2m/a'-2n/b'+p\,(1/a'^2+1/b'^2) = 0 \quad\text{..................(3)},$$
$$1-2m/a-2n/b'+p\,(1/a^2+1/b'^2) = 0 \quad\text{..................(4)}.$$

Ces équations se réduisent à trois, car $(1)+(3) = (2)+(4)$; donc les quatre points α, β, γ, δ sont concycliques.

Si l'on soustrait (1) de (2) et de (4), on obtient

$$m = \tfrac{1}{2}p\,(1/a+1/a'), \quad n = \tfrac{1}{2}p\,(1/b+1/b') \quad\text{..................(5)}.$$

En portant les valeurs (5) dans (1), il vient $1-p\,(1/aa'+1/bb') = 0$; ce qui démontre la première formule proposée. Ensuite

$$\mathrm{R}^2 = m^2+n^2-p = \tfrac{1}{4}p^2\left[(1/a+1/a')^2+(1/b+1/b')^2\right]-p,$$
$$4\mathrm{R}^2/p^2 = (1/a+1/a')^2+(1/b+1/b')^2-4\,(1/aa'+1/bb')$$
$$= (1/a-1/a')^2+(1/b-1/b')^2.$$

Les formules proposées peuvent s'étendre à un octaèdre à trois diagonales rectangulaires concourantes (*Mathesis*, 1904, p. 258).

(II.) Let the angles OAB, OBC, OCD, ODA each equal θ, ϕ, ψ, χ, and, if Ω be the centre of the circle $\alpha\beta\gamma\delta$, let $\angle \Omega OC = \eta$.

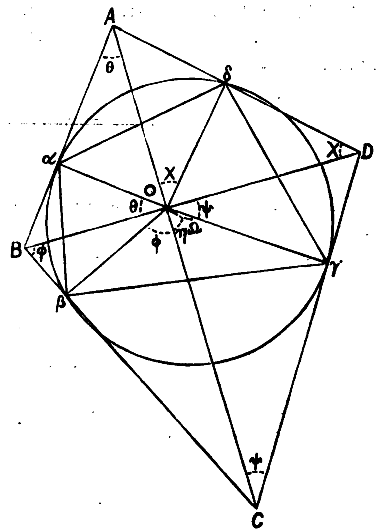
Again, let

$$OA = a, \quad OB = b,$$
$$OC = c, \quad OD = d;$$

then

$$O\alpha = ab/(a^2+b^2)^{\frac{1}{2}},$$
$$O\beta = bc/(b^2+c^2)^{\frac{1}{2}},$$
$$O\gamma = cd/(c^2+d^2)^{\frac{1}{2}},$$
$$O\delta = da/(d^2+a^2)^{\frac{1}{2}},$$

and $\quad p = R^2 - O\Omega^2.$



Hence

$$p = O\alpha^2 + 2O\Omega \,.\, O\alpha \sin(\theta+\eta);$$
$$p = O\beta^2 - 2O\beta \,.\, O\Omega \cos(\phi+\eta);$$
$$p = O\gamma^2 - 2O\gamma \,.\, O\Omega \sin(\psi+\eta);$$
$$p = O\delta^2 + 2O\delta \,.\, O\Omega \cos(\chi+\eta).$$

Now these equations may be written

$$p(a^2+b^2) = a^2b^2 + 2ab \,.\, O\Omega\,(b\cos\eta + a\sin\eta) \quad \text{..............(i.)},$$
$$p(b^2+c^2) = b^2c^2 - 2bc \,.\, O\Omega\,(b\cos\eta - c\sin\eta) \quad \text{..................(ii.)},$$
$$p(c^2+d^2) = c^2d^2 - 2cd \,.\, O\Omega\,(d\cos\eta + c\sin\eta) \quad \text{...............(iii.)},$$
$$p(d^2+a^2) = d^2a^2 + 2da \,.\, O\Omega\,(d\cos\eta - a\sin\eta) \quad \text{..............(iv.)}.$$

Multiplying (i.) and (ii.) by c and a, and adding,

$$p\,[c(a^2+b^2) + a(b^2+c^2)] = abc\,(ab+bc) + 2abc\,(a+c)\sin\eta \,.\, O\Omega \quad \text{...}(\alpha).$$

Similarly, from (iii.) and (iv.),

$$p\,[a(c^2+d^2) + c(d^2+a^2)] = acd\,(cd+da) - 2acd\,(a+c)\sin\eta \,.\, O\Omega \quad \text{...}(\beta).$$

Multiplying (α) and (β) by d and b, and adding,

$$p\,[(a+c)(b+d)(ac+bd)] = abcd\,(a+c)(b+d);$$

therefore $\quad 1/p = (ac+bd)/abcd = 1/OA \,.\, OC + 1/OB \,.\, OD.$

Again, from (i.) and (ii.) by subtraction,

$$p\,(a-c) - b^2(a-c) = 2b^2 \,.\, O\Omega\cos\eta + 2b \,.\, O\Omega \,.\, (a-c)\sin\eta.$$

Similarly, from (iii.) and (iv.),

$$p\,(a-c) - d^2(a-c) = 2d^2 \,.\, O\Omega\cos\eta - 2d \,.\, O\Omega \,.\, (a-c)\sin\eta.$$

From these equations,

$$2bd \,.\, O\Omega\cos\eta = abcd\,(a-c)/(ac+bd) - bd\,(a-c),$$

since $\quad p = abcd/(ac+bd),$

and $\quad 2bd \,.\, O\Omega\sin\eta = abcd\,(d-b)/(ac+bd).$

Squaring each of the last results, and adding,

$$4O\Omega^2 = [b^2d^2(a-c)^2 + a^2c^2(b-d)^2]/(ac+bd)^2;$$

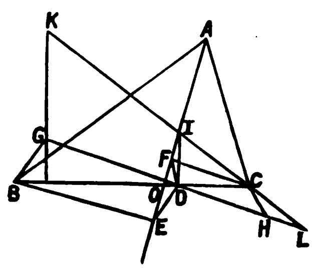
therefore $4R^2/p^2 = (4p + 4O\Omega^2)/p^2$

$$= [4abcd\,(ac+bd) + b^2d^2(a-c)^2 + a^2c^2(b-d)^2]/(ac+bd)^2$$
$$= [(a+c)/ac]^2 + [(b+d)/bd]^2$$
$$= (1/OA + 1/OC)^2 + (1/OB + 1/OD)^2.$$

Note by the PROPOSER.—La divergence des résultats quant aux signes n'est qu'apparente, parce que l'auteur applique la règle des signes aux quantités OA, OB, OC, OD.

9749. (Professor CATALAN.)—ABC étant un triangle donné, soit D le point de contact avec BC du cercle inscrit I. On projette les sommets B, C en E, F sur la bissectrice AO ; puis l'on construit les parallélogrammes DEBG, DFCH. Cela posé (1) les points B, G, C, H appartiennent à une circonférence ; (2) le centre de cette circonférence et le centre I du cercle inscrit sont également distants du côté BC.

Solutions (I.) *by* R. F. DAVIS, M.A., *and* J. A. H. JOHNSTON, M.A. ; (II.) *by* Professor SANJÁNA, M.A., *and* Rev. R. A. THOMAS, M.A. ; (III.) *by* T. DENNIS.



(I.) By construction, GD is parallel to BE, and is therefore perpendicular to AO. Similarly, HD is perpendicular to AO ; and G, D, H are collinear. Since BEDI is cyclic,

$$\angle \mathrm{BDE} = \angle \mathrm{BIE} = \tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \tfrac{1}{2}(\pi - \mathrm{C}),$$

or DE is perpendicular to IC. Similarly, DF is perpendicular to IB.

$$\angle \mathrm{GBC} = \angle \mathrm{BDE} = \angle \mathrm{CID} = \angle \mathrm{CFD} \quad \text{(since CIFD is cyclic)}$$
$$= \angle \mathrm{GHC}\ ;$$

thus BGCH is cyclic.

Let CI meet GH in L and the perpendicular from G on BC in K. Since D is the orthocentre of the triangle IEL, ID is perpendicular to EL. Therefore EL is parallel to BC and BDLE is a parallelogram, so that DL = BE = GD, and KG is consequently = 2ID. Now in the triangle GBC the distance of the circum-centre from the base BC is half the distance of the orthocentre K from the vertex G ; that is, = ID.

(II.) As BE, CF are parallel, DG, DH are in a straight line, and

$$\mathrm{DG}\,.\,\mathrm{DH} = \mathrm{BE}\,.\,\mathrm{CF} = bc\sin^2\tfrac{1}{2}\mathrm{A} = (s-b)(s-c) = \mathrm{BD}\,.\,\mathrm{DC}\ ;$$

so that B, G, C, H are concyclic. Again $\angle$ BDI = $\angle$ BEI ; therefore $\angle$ BDE = $\angle$ BIE ; thus $\angle$ GBC = $\angle$ BDE = $90^\circ - \tfrac{1}{2}$C. Hence the radius (ρ) of the circle BGCH = GC/(2 sin GBC) = GC/(2 cos $\tfrac{1}{2}$C) ; so that

$$4\rho^2\cos^2\tfrac{1}{2}\mathrm{C} = \mathrm{GC}^2 = \mathrm{GD}^2 + \mathrm{DC}^2 + 2\mathrm{GD}\,.\,\mathrm{DC}\cos\mathrm{GDB}$$
$$= c^2\sin^2\tfrac{1}{2}\mathrm{A} + (s-c)^2 + 2c\,(s-c)\sin\tfrac{1}{2}\mathrm{A}\cos\tfrac{1}{2}(\mathrm{C}-\mathrm{B}).$$

Employing the formulæ $r\cos\frac{1}{2}C = (s-c)\sin\frac{1}{2}C = c\sin\frac{1}{2}A\sin\frac{1}{2}B$, we get

$$\frac{r^2\cos^2\frac{1}{2}C}{\sin^2\frac{1}{2}B}+\frac{r^2\cos^2\frac{1}{2}C}{\sin^2\frac{1}{2}C}+2\frac{r^2\cos^2\frac{1}{2}C}{\sin\frac{1}{2}B\sin\frac{1}{2}C}\cos\tfrac{1}{2}(C-B)$$

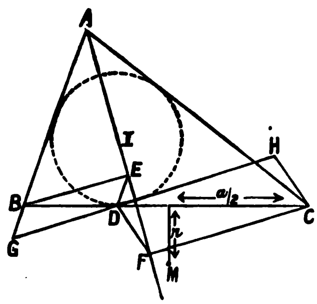
$$=\frac{r^2\cos^2\frac{1}{2}C}{\sin^2\frac{1}{2}B\sin^2\frac{1}{2}C}\left[\sin^2\tfrac{1}{2}C+\sin^2\tfrac{1}{2}B+2\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\,(\cos\tfrac{1}{2}C\cos\tfrac{1}{2}B+\sin\tfrac{1}{2}C\sin\tfrac{1}{2}B)\right]$$

$$=\frac{r^2\cos^2\frac{1}{2}C}{\sin^2\frac{1}{2}B\sin^2\frac{1}{2}C}\left(\sin^2\tfrac{1}{2}C-\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C+\sin^2\tfrac{1}{2}B-\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C+2\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C\cos\tfrac{1}{2}B+4\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C\right)$$

$$=\frac{r^2\cos^2\frac{1}{2}C}{\sin^2\frac{1}{2}B\sin^2\frac{1}{2}C}\left[(\sin\tfrac{1}{2}C\cos\tfrac{1}{2}B+\cos\tfrac{1}{2}C\sin\tfrac{1}{2}B)^2+4\sin^2\tfrac{1}{2}B\sin^2\tfrac{1}{2}C\right]$$

$$=\frac{r^2\cos^2\frac{1}{2}C\cos^2 A}{\sin^2\frac{1}{2}B\sin^2\frac{1}{2}C}+4r^2\cos^2\tfrac{1}{2}C.$$

Thus, finally, $4\rho^2 = a^2+4r^2$; therefore $\rho^2 = r^2+(\frac{1}{2}a)^2$. As the centre is in the perpendicular bisector of BC, we see that its distance from BC equals r, *i.e.*, ID.



(III.) 1. Let r be the radius; then

$$\text{BD}\,.\,\text{DC} = r^2\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C$$
$$= a^2\sin B\sin C/4\cos^2\tfrac{1}{2}A$$
$$= bc\sin^2\tfrac{1}{2}A = \text{BE}\,.\,\text{CF}$$
$$= \text{GD}\,.\,\text{DH};$$

therefore B, G, C, H are concyclic.

2. Let M be on the perpendicular bisector of BC and at a distance r from it. Then

$$\text{MC}^2 = r^2+\tfrac{1}{4}a^2,$$

$$\text{MH}^2 = [r+b\sin\tfrac{1}{2}A\cos(C+\tfrac{1}{2}A)]^2+[-\tfrac{1}{2}a+b\sin\tfrac{1}{2}A\sin(C+\tfrac{1}{2}A)+r\cot\tfrac{1}{2}B]^2;$$

therefore

$$\text{MH}^2-\text{MC}^2 = b^2\sin^2\tfrac{1}{2}A+r^2\cot^2\tfrac{1}{2}B+2br(\sin\tfrac{1}{2}A/\sin\tfrac{1}{2}B)[\sin\tfrac{1}{2}B\cos(C+\tfrac{1}{2}A)+\sin(C+\tfrac{1}{2}A)\cos\tfrac{1}{2}B]-ar\cot\tfrac{1}{2}B-ab\sin\tfrac{1}{2}A\sin(C+\tfrac{1}{2}A)$$

$$= -r^2\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C+2br\sin\tfrac{1}{2}A\cos\tfrac{1}{2}C/\sin\tfrac{1}{2}B+b^2\sin^2\tfrac{1}{2}A-ab\sin\tfrac{1}{2}A\sin(C+\tfrac{1}{2}A).$$

But $$r = a\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C/\cos\tfrac{1}{2}A;$$

therefore

$$\text{MH}^2-\text{MC}^2 = -a^2\sin B\sin C/4\cos^2\tfrac{1}{2}A+2ab\sin\tfrac{1}{2}C\sin\tfrac{1}{2}A\cos\tfrac{1}{2}C/\cos\tfrac{1}{2}A+a^2\sin^2 B/4\cos^2\tfrac{1}{2}A-ab\sin\tfrac{1}{2}A\sin(C+\tfrac{1}{2}A)$$

$$= a^2(\sin B/4\cos^2\tfrac{1}{2}A)[-\sin C+\sin B+2\sin C-(\sin B+\sin C)]$$

$$= 0;$$

therefore MH = MC, and M is the required centre of the circle BGCH.

15704. (R. F. Whitehead, B.A.)—Expand $\theta/\sin\theta$ in ascending powers of $\cos\theta$.

Solution by the PROPOSER.

$$\frac{\theta}{\sin\theta} = \int_0^{\frac{1}{2}\pi} \frac{dx}{1+\cos\theta\cos x} = \int_0^{\frac{1}{2}\pi} dx\,(1-\cos\theta\cos x+\cos^2\theta\cos^2 x-\ldots)$$

$$= \tfrac{1}{2}\pi\left(1+\tfrac{1}{2}\cos^2\theta+\frac{1\,.\,3}{2\,.\,4}\cos^4\theta+\ldots\right) - \left(\cos\theta+\tfrac{2}{3}\cos^3\theta+\frac{2\,.\,4}{3\,.\,5}\cos^5\theta+\ldots\right).$$

15699. (JAMES BLAIKIE, M.A.)—Prove that $m^{2n+1}+(m-1)^{n+2}$ is a multiple of m^2-m+1. [*E.g.*, $1000^{15}+999^9 = \text{M}\,(999001)$.]

Additional Solution by R. W. D. CHRISTIE.

We have $(m+1)(m^2-m+1) = m^3+1 = 0$, mod (m^2-m+1);

therefore $m^{2n+1}(m^3+1) = m^{2n+4}+m^{2n+1} = 0$ mod (m^2-m+1) ....(1).

Again $m^2 = (m-1)$ mod (m^2-m+1);

thus $m^{2n+4} = (m-1)^{n+2}$ mod (m^2-m+1) ....(2);

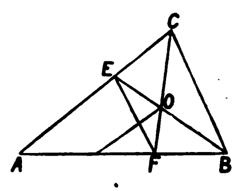
therefore (1) becomes $(m-1)^{n+2}+m^{2n+1}$. Similarly its cognate

$$m^{2n+1} \mp (m+1)^{n+2} = 0 \text{ mod } (m^2+m+1),$$

the sign plus or minus according as n is odd or even.

15105. (SARODA PROSAD BAUERJEE.)—From a point O, taken at random in a triangle ABC, the lines BO and CO are drawn to meet the opposite sides in E and F. Find the mean area of the circle circumscribed about AEF.

Solution by the PROPOSER.



Let us take AB and AC for the axes of x and y; and, as usual, let AB $= c$ and AC $= b$. If x', y' be the co-ordinates of O, the equation of COF, which joins $(0, b)$ and (x', y'), is

$$x(y'-b)-x'y+bx' = 0.$$

Putting $y = 0$, we get $\text{AF} = \dfrac{bx'}{b-y'}$.

Similarly, $\text{AE} = \dfrac{cy'}{c-x'}$. Now,

$$\text{EF}^2 = \text{AF}^2+\text{AE}^2-2\text{AF}\,.\,\text{AE}\cos\text{A}$$

$$= \frac{b^2x'^2}{(b-y')^2}+\frac{c^2y'^2}{(c-x')^2}-2bc\cos\text{A}\,\frac{x'y'}{(c-x')(b-y')}.$$

And, if R be the radius of the circle circumscribed about AEF, and S its area, we have $\text{R} = \text{EF}/(2\sin\text{A})$. Therefore

$$\text{S} = \pi\text{R}^2 = \frac{\pi}{4\sin^2\text{A}}\,\text{EF}^2.$$

Hence

$$M(S) = \frac{1}{\Delta ABC}\frac{\pi}{4\sin^2 A}$$

$$\times \iint \left\{ \frac{b^2x'^2}{(b-y')^2} + \frac{c^2y'^2}{(c-x')^2} - 2bc\cos A \frac{x'y'}{(c-x')(b-y')} \right\} \sin A\, dx'dy$$

$$= \frac{\pi}{2bc\sin^2 A} \iint \left\{ \frac{b^2x'^2}{(b-y')^2} + \frac{c^2y'^2}{(c-x')^2} - 2bc\cos A \frac{x'y'}{(c-x')(b-y')} \right\} dx'dy'.$$

Now, since the integral is to extend over the entire area of the triangle ABC, the limits of y' are from 0 to $b(1-x'/c)$, and that of x' from 0 to c.

Transforming the integral by putting $x' = cu$ and $y' = bv$, we get $dx'dy' = bc\,du\,dv$, and the limits of v are from 0 to $1-u$ and those of u from 0 to 1. Hence

$$M(S) = \frac{\pi}{2\sin^2 A}\int_0^1\int_0^{1-u} \left\{ \frac{c^2u^2}{(1-v)^2} + \frac{b^2v^2}{(1-u)^2} - 2bc\cos A \frac{uv}{(1-u)(1-v)} \right\} du\,dv$$

$$= \frac{\pi}{2\sin^2 A}\left\{ c^2\int_0^1\int_0^{1-u}\frac{u^2\,du\,dv}{(1-v)^2} + b^2\int_0^1\int_0^{1-u}\frac{v^2\,du\,dv}{(1-u)^2} \right.$$

$$\left. - bc\cos A\int_0^1\int_0^{1-u}\frac{2uv\,du\,dv}{(1-u)(1-v)} \right\}.$$

Now the first two integrals can easily be seen to be each equal to $\frac{1}{6}$. The last integral, namely,

$$\int_0^1\int_0^{1-u}\frac{2uv\,du\,dv}{(1-u)(1-v)} = \int_0^1\frac{2u\,du}{1-u}\int_0^{1-u}\frac{v\,dv}{1-v} = \int_0^1\frac{2u\,du}{1-u}\int_0^{1-u}\left(\frac{dv}{1-v} - dv\right)$$

$$= \int_0^1 (u-1-\log u)\frac{2u\,du}{1-u} = -\int_0^1 2u\,du - 2\int_0^1\frac{u\log u}{1-u}du$$

$$= -1 - 2\int_0^1\frac{1-(1-u)}{1-u}\log u\,du$$

$$= -1 - 2\int_0^1\frac{\log u\,du}{1-u} + 2\int_0^1\log u\,du$$

$$= -1 + 2\cdot\tfrac{1}{6}\pi^2 - 2 = \tfrac{1}{3}\pi^2 - 3.$$

Hence
$$M(S) = \frac{\pi}{2\sin^2 A}\left\{\tfrac{1}{6}b^2 + \tfrac{1}{6}c^2 - bc\cos A\left(\tfrac{1}{3}\pi^2 - 3\right)\right\}$$

$$= \frac{\pi}{12\sin^2 A}\left\{b^2 + c^2 - 2bc\cos A(\pi^2 - 9)\right\}.$$

15724. (S. C. Bagchi, B.A.)—The transformation

$$\xi = \alpha x + \beta y, \quad \eta = \gamma x + \delta y$$

makes a point (ξ, η) on the curve c_2 correspond to a point (x, y) on the curve c_1. If $\alpha\delta - \beta\gamma = \pm 1$, show that, when c_1 satisfies $\phi(p^3\rho) = 0$, c_2 will satisfy the same functional equation, where ρ is the radius of curvature at a point and p is the distance of the tangent at that point from the origin.

Solution by the PROPOSER.

The correspondence is obtained by linear substitutions for ξ and η. The operators of the first and second degree are respectively

$$\alpha\,(\partial/\partial\xi)+\beta\,(\partial/\partial\eta),\ \ldots \quad \text{and} \quad \alpha^2\,(\partial^2/\partial\xi^2)+2\alpha\beta\,(\partial^2/\partial\xi\,\partial\eta)+\beta^2\,(\partial^2/\partial\eta^2),\ \ldots .$$

Now on making these operators perform their functions on the given form it is at once found that the new shape of $\phi\,(p^3\rho)=0$ differs from the original by the presence of the extraneous factor $(\alpha\delta-\beta\gamma)^2$. Therefore the given theorem follows.

15739. (Professor SANJÁNA, M.A.)—Evaluate

$$\left[(\tanh ax)^{-2}-(ax-\tfrac{1}{3}a^3x^3)^{-2}\right]/x^2$$

when $x=0$.

Solution by C. M. ROSS.

$$\tanh ax=(\epsilon^{ax}-\epsilon^{-ax})/(\epsilon^{ax}+\epsilon^{-ax})=ax\left(1+\frac{a^2x^2}{3!}+\ldots\right)\Big/\left(1+\frac{a^2x^2}{2!}+\ldots\right).$$

Hence expression becomes

$$\left[\left\{\left(1+\frac{a^2x^2}{2!}+\ldots\right)^2\left(1+\frac{a^2x^2}{3!}+\ldots\right)^{-2}\right\}\Big/a^2x^2-\left\{1-\tfrac{1}{3}a^2x^2\right\}^{-2}/a^2x^2\right]\Big/x^2$$

$$=\left\{\left(1+\frac{2a^2x^2}{2!}+\frac{2a^4x^4}{4!}\ldots\right)\left(1-\frac{2a^2x^2}{3!}-\frac{2a^4x^4}{5!}\ldots\right)\Big/a^2x^2\right.$$
$$\left.-(1+\tfrac{2}{3}a^2x^2+\ldots)/a^2x^2\right\}\Big/x^2$$

$$=\left\{-a^2x^2\left(\frac{2}{3!}-\frac{2}{2!}+\frac{2}{3}\right)-\left(\frac{2}{5!}+\frac{4}{2!\,3!}-\frac{2}{4!}\ldots\right)a^4x^4\right\}\Big/a^2x^4$$

$$=-\left(\frac{2}{5!}+\frac{4}{2!\,3!}-\frac{2}{4!}\right)a^2+\text{higher powers of }x^2\ldots=-\tfrac{4}{15}a^2,$$

since higher powers of x^2 vanish in the limit.

15693. (Professor NANSON.)—If a fixed straight line cut one of a series of concentric similar and similarly situated conics at angles θ, ϕ, the length of the intercepted chord varies inversely as $\cot\theta+\cot\phi$.

Solution by Professor SANJÁNA, M.A.

The conics will be represented by $x^2/a^2+y^2/a^2k^2=1$, where a is variable and k is constant $(=1-e^2)$. Let the extremities of the chord in any one conic have eccentric angles α and β. The tangents at these points are $x\cos\alpha/a+y\sin\alpha/ak=1$, $x\cos\beta/a+y\sin\beta/ak=1$, and the chord is $x\cos\tfrac{1}{2}(\alpha+\beta)/a+y\sin\tfrac{1}{2}(\alpha+\beta)/ak=\cos\tfrac{1}{2}(\alpha-\beta)$. Hence $\cot\theta$ is

$$[1+k^2\cot\alpha\cot\tfrac{1}{2}(\alpha+\beta)]/[k\cot\tfrac{1}{2}(\alpha+\beta)-k\cot\alpha]$$
$$=[\sin\alpha\sin\tfrac{1}{2}(\alpha+\beta)+k^2\cos\alpha\cos\tfrac{1}{2}(\alpha+\beta)]/[k\sin\tfrac{1}{2}(\alpha-\beta)];$$

so $\cot\phi = \sin\beta \sin\frac{1}{2}(\alpha+\beta) + k^2 \cos\beta \cos\frac{1}{2}(\alpha+\beta)/[k \sin\frac{1}{2}(\alpha-\beta)]$.

Thus

$\cot\theta + \cot\phi$

$= [\sin\frac{1}{2}(\alpha+\beta)(\sin\alpha+\sin\beta) + k^2\cos\frac{1}{2}(\alpha+\beta)(\cos\alpha+\cos\beta)]/[k\sin\frac{1}{2}(\alpha-\beta)]$

$= 2\cos\frac{1}{2}(\alpha-\beta)[\sin^2\frac{1}{2}(\alpha+\beta) + k^2\cos^2\frac{1}{2}(\alpha+\beta)]/[k\sin\frac{1}{2}(\alpha-\beta)]$;

also the square of L, the length of the chord, is

$$a^2(\cos\alpha-\cos\beta)^2 + a^2k^2(\sin\alpha-\sin\beta)^2 = 4a^2\sin^2\tfrac{1}{2}(\alpha-\beta)[\sin^2\tfrac{1}{2}(\alpha+\beta) + k^2\cos^2\tfrac{1}{2}(\alpha+\beta)].$$

Therefore

$$L^2(\cot\theta+\cot\phi)^2 = 16a^2\cos^2\tfrac{1}{2}(\alpha-\beta)[\sin^2\tfrac{1}{2}(\alpha+\beta) + k^2\cos^2\tfrac{1}{2}(\alpha+\beta)]^3/k^2.$$

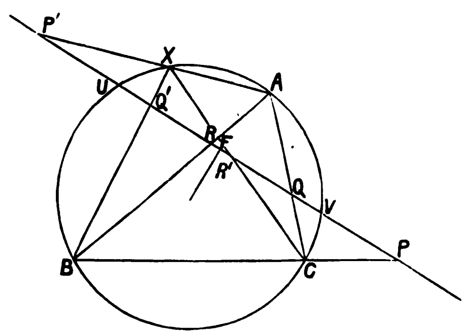
Now, as the chord is a fixed straight line, the ratios of the coefficients in its equation are fixed; so that $k\cot\frac{1}{2}(\alpha+\beta)$ and $ak\cos\frac{1}{2}(\alpha-\beta)/\sin\frac{1}{2}(\alpha+\beta)$ are fixed. From the former we get $\frac{1}{2}(\alpha+\beta)$ constant, and then the latter gives $a\cos\frac{1}{2}(\alpha-\beta)$ also constant. Thus $L^2(\cot\theta+\cot\phi)^2$ is fixed, and therefore L varies inversely as $\cot\theta+\cot\phi$.

15780. (JAMES BLAIKIE, M.A.)—ABC is a triangle of which O is the circum-centre, and BC, CA, AB meet a given straight line in P, Q, R; F is the foot of the perpendicular from O to the given line; and P′, Q′, R′ are points in the line such that F is the mid-point of PP′, QQ′, RR′. Prove that AP′, BQ′, CR′ meet in a point on the circum-circle of ABC.

Another Solution by Professor SANJÁNA, M.A.

Since F is the mid-point of UV, P and P′ are isotomically conjugate to U and V; so are Q and Q′, R and R′. Hence AP′, BQ′, CR′ are concurrent. (See Question 14233, *Reprint*, Vol. LXXV., p. 28.)

Again (PP′, QQ′, RR′) is an involution of which U and V, F and ∞, are conjugate points: hence, the connectors of the quadrilateral ACBX



and the circle ACB cut the line UV in eight points in involution. Thus X also lies upon the circle. (Converse of § 273, Lachlan, *Modern Pure Geometry*.)

The PROPOSER wishes to point out that this is the more general theorem of which Question 15691 is a particular case.

[N.B.—For solution by the PROPOSER of the present problem, see *Reprint*, Vol. VIII., N.S., p. 76 (now in press).]

15747. (J. L. S. HATTON.)—If

$$ax^2 + by^2 + cz^2 + 2hxy + 2gzx + 2fyz = 0$$

be the general equation of the second degree in trilinear co-ordinates, show that the necessary and sufficient condition that it should represent a circle is

$$(a+b+c-2h\cos C-2g\cos B-2f\cos A)^2+4\begin{vmatrix} a & h & g & \sin A \\ h & b & f & \sin B \\ g & f & c & \sin C \\ \sin A & \sin B & \sin C & 0 \end{vmatrix} = 0.$$

Solutions (I.) *by* J. A. H. JOHNSTON, M.A.; (II.) *by* Professor SANJÁNA, M.A.; (III.) *by* C. M. ROSS.

(I.) The necessary and sufficient condition for a circle in Cartesians is that $(a-b)^2 + 4h^2 = 0$, implying $a = b$ and $h = 0$. This may be written

$$(a+b)^2 + 4(h^2 - ab) = 0 \quad \text{.........................(i.).}$$

Now $(a+b)$ and $(ab-h^2)$ for all linear transformations have associated invariable forms, *e.g.*, if

$$X = x\cos\phi + y\sin\phi - p_1, \quad Y = x\cos\chi + y\sin\chi - p_2,$$
$$Z = x\cos\psi + y\sin\psi - p_3,$$

and if these values be substituted in the general equation

$$aX^2 + bY^2 + cZ^2 + 2fYZ + 2gZX + 2hXY = 0,$$

it is at once demonstrable that $(a+b)$ transforms into

$$(a+b+c-2f\cos A-2g\cos B-2h\cos C),$$

and $(ab-h^2)$ into $A\sin^2 A + B\sin^2 B + C\sin^2 C + 2F\sin B\sin C$
$+ 2G\sin C\sin A + 2H\sin A\sin B,$

where $\chi - \psi = 180^\circ - A$, &c.

The condition for a circle in trilinears may therefore be written by (i.),

$$(a+b+c-2f\cos A-2g\cos B-2h\cos C)^2 - 4(A\sin^2 A + \ldots) = 0,$$

which is the required result.

This result is given in Salmon's *Conic Sections*, p. 352, sixth edition, and proved in another way.

(II.) The equations for determining the lengths of the principal axes of a conic are

$$AB = (abc/4\Delta^2)\,[H/(-K)^{\frac{3}{2}}], \quad A^2 + B^2 = -(a^2b^2c^2/16\Delta^2)\,(EH/K^2)$$

(WHITWORTH, *Modern An. Geo.*, § 285). For the conic to be a circle, the

necessary and sufficient condition is $A = B$; hence

$$-(a^2b^2c^2/16\Delta^2)(EH/K^2) = (abc/2\Delta^2)[H/(-K)^{\frac{1}{2}}].$$

From this we get $-abcE = 8\Delta(-K)^{\frac{1}{2}}$, *i.e.*, $a^2b^2c^2E^2 + 64\Delta^2K = 0$. Here

$$E \equiv a+b+c-2f\cos A - 2g\cos B - 2h\cos C,$$

and K is the bordered discriminant $\begin{vmatrix} a & h & g & f' \\ h & b & f & g' \\ g & f & c & h' \\ f' & g' & h' & 0 \end{vmatrix}$; in the latter

$f' = a/2\Delta = \sin A\,(abc/4\Delta^2)$, $g' = \sin B\,(abc/4\Delta^2)$, $h' = \sin C\,(abc/4\Delta^2)$.
Substituting, simplifying, and cancelling $a^2b^2c^2$, we get

$$(a+b+c-2f\cos A - 2g\cos B - 2h\cos C)^2 + 4\begin{vmatrix} a & h & g & \sin A \\ h & b & f & \sin B \\ g & f & c & \sin C \\ \sin A & \sin B & \sin C & 0 \end{vmatrix} = 0.$$

(III.) It is known that the trilinear equation of any circle is

$$yz\sin A + zx\sin B + xy\sin C + (lx+my+nz)(x\sin A + y\sin B + z\sin C) = 0.$$

Comparing the coefficients of this equation with the given one,

$$Ka = l\sin A,\quad Kb = m\sin B,\quad Kc = n\sin C \;\ldots\ldots(1, 2, 3),$$

$$2Kf = \sin A + m\sin C + n\sin B,\quad 2Kg = \sin B + n\sin A + l\sin C \ldots(4, 5),$$

$$2Kh = l\sin B + m\sin A + \sin C \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

K being a constant. Now, multiplying (4), (5), and (6) respectively by $\cos A \cos B \cos C$ and adding,

$$2K(h\cos C + g\cos B + f\cos A) = l\sin A + m\sin B + n\sin C + 2\sin A\sin B\sin C.$$

But, adding the equations (1), (2), and (3),

$$l\sin A + m\sin B + n\sin C = K(a+b+c);$$

therefore

$$(a+b+c-2h\cos C - 2g\cos B - 2f\cos A)^2 K^2 = 4\sin^2 A\sin^2 B\sin^2 C \ldots(A).$$

Again,

$$4\Sigma(f^2-bc)K^2 = \Sigma(\sin A + m\sin C - n\sin B)^2,$$

$$8\Sigma(af-gh)K^2 = \Sigma\{2l\sin A(\sin A + m\sin C + n\sin B) - \tfrac{1}{2}(\sin B + n\sin A + l\sin C)(l\sin B + m\sin A + \sin C)\};$$

therefore

$$4\{\Sigma(f^2-bc)\sin^2 A - \Sigma(af-gh)\sin B\sin C\}K^2 = -4\sin^2 A\sin^2 B\sin^2 C,$$

by simplification. Now the sinister side of this equation is equivalent to the determinant $4\begin{vmatrix} a & h & g & \sin A \\ h & b & f & \sin B \\ g & f & c & \sin C \\ \sin A & \sin B & \sin C & 0 \end{vmatrix}$; therefore the required condition becomes

$$(a+b+c-2h\cos C - 2g\cos B - 2f\cos A)^2 + 4\begin{vmatrix} a & h & g & \sin A \\ h & b & f & \sin B \\ g & f & c & \sin C \\ \sin A & \sin B & \sin C & 0 \end{vmatrix} = 0.$$

Note.—The PROPOSER remarks that Solution (III.) establishes the necessity of the conditions, but does not seem to prove clearly its sufficiency.

15756. (Professor NEUBERG.)—Trouver $\int_0 \cos x \cos 2x \cos 3x \cos 4x \, dx$.

Solution by C. M. Ross.

$$\begin{aligned}\cos x \cos 2x \cos 3x \cos 4x &= \tfrac{1}{4}(\cos 4x + \cos 2x)(\cos 6x + \cos 2x)\\ &= \tfrac{1}{4}(\cos 4x \cos 6x + \cos 2x \cos 4x + \cos 2x \cos 6x + \cos^2 2x)\\ &= \tfrac{1}{8}(1 + 2\cos 2x + 2\cos 4x + \cos 6x + \cos 8x + \cos 10x).\end{aligned}$$

Hence

$$\begin{aligned}&\int \cos x \cos 2x \cos 3x \cos 4x \, dx\\ &= \tfrac{1}{8}\int dx + \tfrac{1}{4}\int \cos 2x \, dx + \tfrac{1}{4}\int \cos 4x \, dx + \tfrac{1}{8}\int \cos 6x \, dx + \tfrac{1}{8}\int \cos 8x \, dx + \tfrac{1}{8}\int \cos 10x \, dx\\ &= \tfrac{1}{8}x + \tfrac{1}{8}\sin 2x + \tfrac{1}{16}\sin 4x + \tfrac{1}{48}\sin 6x + \tfrac{1}{64}\sin 8x + \tfrac{1}{80}\sin 10x.\end{aligned}$$

Hence, between the given limits, expression on the dexter becomes ·1998.

N.B.—The angles are taken to be in radian measure.

The Integration of $\int \dfrac{dx}{a \sec x + b \operatorname{cosec} x}$.

By GEORGE SCOTT, M.A.

$$\int \frac{dx}{a \sec x + b \operatorname{cosec} x} = \int \frac{\sin x \cos x \, dx}{a \sin x + b \cos x} = f(x).$$

Put $a = r\cos\alpha$, $b = r\sin\alpha$; then

$$f(x) = \frac{1}{r}\int \frac{\sin x \cos x}{\sin(x+\alpha)}\, dx,$$

$$[\sin(x+\alpha) + \sin(x-\alpha)][\sin(x+\alpha) - \sin(x-\alpha)] = 4 \sin\alpha\cos\alpha\sin x\cos x,$$

or

$$\sin x \cos x = \frac{\sin^2(x+\alpha) - \sin^2(x-\alpha)}{2\sin 2\alpha}.$$

Hence

$$f(x) = \frac{1}{2r\sin 2\alpha}\left[\int \sin(x+\alpha)\,dx - \int \frac{\sin^2(x-\alpha)\,dx}{\sin(x+\alpha)}\right],$$

$$\sin(x-\alpha) = \sin[(x+\alpha) - 2\alpha] = \sin(x+\alpha)\cos 2\alpha - \sin 2\alpha\cos(x+\alpha);$$

therefore

$$\begin{aligned}&\int \frac{\sin^2(x-\alpha)}{\sin(x+\alpha)}\,dx\\ &= \int\left\{\cos^2 2\alpha \sin(x+\alpha) - \sin 4\alpha\cos(x+\alpha) + \frac{[1-\sin^2(x+\alpha)]\sin^2 2\alpha}{\sin(x+\alpha)}\right\}dx\\ &= \int\left[\cos 4\alpha\sin(x+\alpha) - \sin 4\alpha\cos(x+\alpha) + \frac{\sin^2 2\alpha}{\sin(x+\alpha)}\right]dx\end{aligned}$$

or

$$= \int\left[\sin(x-3\alpha) + \frac{\sin^2 2\alpha}{\sin(x+\alpha)}\right]dx\,;$$

therefore

$$f(x) = \frac{1}{2r\sin 2\alpha}\left[\int \sin(x+\alpha)\,dx - \int \sin(x-3\alpha)\,dx - \sin^2 2\alpha \int \frac{dx}{\sin(x+\alpha)}\right]$$

or

$$\begin{aligned}f(x) &= \frac{1}{2r\sin 2\alpha}[\cos(x-3\alpha) - \cos(x+\alpha) - \sin^2 2\alpha \log\tan\tfrac{1}{2}(x+\alpha)]\\ &= 1/r\,[\sin(x-\alpha) - \sin 2\alpha \log\tan^{\frac{1}{2}}\tfrac{1}{2}(x+\alpha)].\end{aligned}$$

where $r = \surd(a^2+b^2)$, $\alpha = \tan^{-1}(b/a)$.

10582. (R. Knowles, B.A.)—The equation to a conic referred to a tangent and normal as axes being $ax^2+bxy+cy^2+gy=0$, prove that, e being the eccentricity, the equation to its transverse axis is

$$[4a(a+c)-(2-e^2)(b^2-4ac)]\,x+2b(a+c)\,y+bg(2-e^2)=0.$$

[N.B.—For (b^2-4ac) read $(4ac-b^2)$.—Ed.]

Another Solution by W. F. Beard, M.A.

Let the equation of a conic referred to its centre be $\alpha x^2+2hxy+\beta y^2=1$; let e be its eccentricity; then

$$(e^2-1)/(2-e^2)^2=(h^2-\alpha\beta)/(\alpha+\beta)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also the lengths and equations of the axes are given by

$$1/r^4-(\alpha+\beta)/r^2+\alpha\beta-h^2=0 \quad\text{and}\quad (\alpha-1/r^2)\,x+hy=0\ldots (2),(3)$$

(see Smith's *Conics*). From (1) and (2),

$$1/r^4-(\alpha+\beta)/r^2+[(1-e^2)(\alpha+\beta)^2]/(2-e^2)^2=0;$$

therefore $\quad 1/r^2=(\alpha+\beta)/(2-e^2) \quad$ or $\quad [(1-e^2)(\alpha+\beta)]/(2-e^2)$.

The second value is easily seen to give the transverse axis. Thus its equation is, from (3), $\{\alpha-[(1-e^2)(\alpha+\beta)]/(2-e^2)\}\,x+hy=0$; or, from (1), $[\alpha(\alpha+\beta)-(2-e^2)(\alpha\beta-h^2)]\,x+h(\alpha+\beta)\,y=0$. The equation of the given conic referred to its centre is $ax^2+bxy+cy^2=2ag^2/(4ac-b^2)$, and the co-ordinates of its centre are $bg/(4ac-b^2)$, $-2ag/(4ac-b^2)$. Thus the equation of its transverse axis is

$$[4a(a+c)-(2-e^2)(4ac-b^2)]\,[x-bg/(4ac-b^2)] + 2b(a+c)\,[y+2ag/(4ac-b^2)]=0$$

or $\qquad [4a(a+c)-(2-e^2)(4ac-b^2)]\,x+2b(a+c)\,y+bg(2-e^2)=0.$

15759. (R. Chartres.)—Find the mean value of the $(2n)$-th power of the area, the perimeter $(2s)$ being constant, (1) of a triangle, (2) of a cyclic quadrilateral. Elementary proof wanted.

Solution by the Proposer.

$\mathrm{M}(\Delta^{2n})=s^n\,\mathrm{M}(x^ny^nz^n)$, where $x=s-a$, $y=s-b$, $z=s-c$. Now $s^{3n}=(x+y+z)^{3n}$ has $(3n+2)!/[2\,(3n)!]$ terms, and the coefficient of $x^ny^nz^n=(3n)!/(n!)^3$; therefore

$$\mathrm{M}(\Delta^{2n})=[s^{4n}\,2\,(n!)^3]/(3n+2)! \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

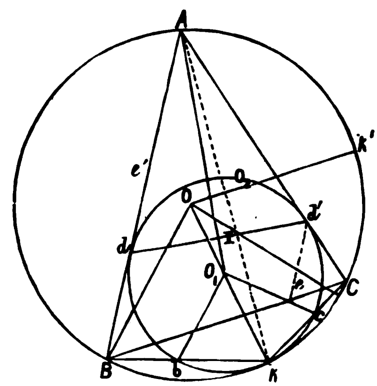
Similarly for a cyclic quadrilateral we have

$$[6\,(2s)^{4n}\,(n!)^4]/(4n+3)! \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If we express (1) and (2) as gamma functions, and then for $2n$ write n, we have the result for all values.

15729. (R. Tucker, M.A.)—The circle dd'K touches AB, AC, and the arc BC of the circle ABC (internally); the circle ee'K′ touches BC, BA, and the arc CA; and the circle ff'K″ touches CA, CB, and the arc AB. Prove (i.) ed', fe', df' are parallel to AB, BC, CA respectively; (ii.) Cd'. Ae'. Bf' = Af. Bd. Ce; (iii.) ρ_1 (radius of circle dd'K) = $r \sec^2 \frac{1}{2}$A; (iv.) AK, BK′, CK″ intersect in a point.

Solution by Professor Sanjána, M.A.



It is known that the radius of dd'K is $r \sec^2 \frac{1}{2}$A, and that its polar for A goes through I, the in-centre; see Vols. LXVI. (p. 46) and LXV. (p. 27). Hence

$$Ad' = AI \sec \tfrac{1}{2}A = r \operatorname{cosec} \tfrac{1}{2}A \sec \tfrac{1}{2}A = 2r/\sin A;$$

So also $Be' = 2r/\sin B$.

Thus $Ad'/Be = \sin B/\sin A = AC/BC$,

i.e., $d'e$ is parallel to AB; so also for $e'f$, $f'd$. Again

$$Cd' = b - 2r/\sin A,$$

$$Ae' = c - 2r/\sin B, \quad Bf' = a - 2r/\sin C;$$

so for Af, Bd, Ce, and it can readily be proved that

$$(a - 2r/\sin C)(b - 2r/\sin A)(c - 2r/\sin B) = (b - 2r/\sin C)(c - 2r/\sin A)(a - 2r/\sin B),$$

whence the second result.

Lastly, join KB, KC, cutting the inner circle in b, c; then, since K is a centre of similitude of the two circles, KB/KC = Kb/Kc = Bb/Cc, so that

$$KB^2/KC^2 = (KB.Bb)/(KC.Cc) = Bd^2/Cd'^2.$$

Thus
$$\frac{KB}{KC} = \frac{Bd}{Cd'} = \frac{c - 2r/\sin A}{b - 2r/\sin A} = \frac{c \sin A - 2r}{b \sin A - 2r} = \frac{2\Delta/b - 2\Delta/s}{2\Delta/c - 2\Delta/s} = \frac{(s-b)/b}{(s-c)/c}$$

$$= \frac{(s-a)(s-b)}{ab} \div \frac{(s-c)(s-a)}{ca} = \sin^2 \tfrac{1}{2}C/\sin^2 \tfrac{1}{2}B.$$

Now
$$\frac{\sin BAK}{\sin CAK} = \frac{2R \sin BAK}{2R \sin CAK} = \frac{BK}{CK} = \frac{\sin^2 \frac{1}{2}C}{\sin^2 \frac{1}{2}B};$$

so also for the other angles. Therefore

$$\frac{\sin BAK \sin CBK' \sin ACK''}{\sin CAK \sin ABK' \sin BCK''} = 1,$$

and hence AK, BK′, CK″ cointersect.

[*Note.*—In the diagram read K and K′ for k and k'.—Ed.]

15768. (Professor Sanjána, M.A.)—If P, Q be any two points isogonally conjugate with regard to the triangle ABC, then

$$(AP.AQ)/(AB.AC) + (BP.BQ)/(BC.BA) + (CP.CQ)/(CA.CB) = 1.$$

Hence prove that

(i.) $a \,.\, m_1 \,.\, \mathrm{AK} + b \,.\, m_2 \,.\, \mathrm{BK} + c \,.\, m_3 \,.\, \mathrm{CK} = \frac{3}{2}abc,$

where m_1, m_2, m_3 are the medians ~~and K~~ is the symmedian point;

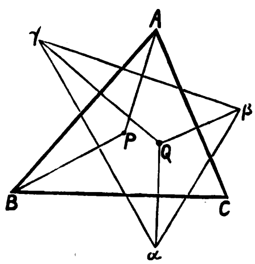
(ii.) $a \,.\, \mathrm{AI}^2 + b \,.\, \mathrm{BI}^2 + c \,.\, \mathrm{CI}^2 = a \,.\, \mathrm{AI}_1{}^2 - b \,.\, \mathrm{BI}_1{}^2 - c \,.\, \mathrm{CI}_1{}^2 = abc,$

where I is the in-centre and I_1 the first ex-centre;

(iii.) $a \,.\, \mathrm{AP} + b \,.\, \mathrm{BP} + c \,.\, \mathrm{CP} = abc/\mathrm{R},$

where P is the orthocentre.

Solutions (I.) *by* R. F. Davis, M.A.; (II.) *by* M. V. A. Sastry, B.A.



(I.) Let α, β, γ be the images of Q in BC, CA, AB respectively. Then

$$\mathrm{A}\beta = \mathrm{AQ} = \mathrm{A}\gamma;$$

also, since $\beta\mathrm{AC} = \mathrm{QAC} = \mathrm{PAB}$

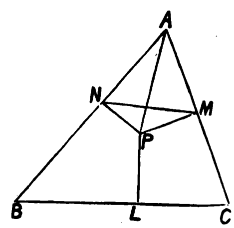
and $\gamma\mathrm{AB} = \mathrm{QAB} = \mathrm{PAC},$

therefore $\mathrm{PA}\beta = \mathrm{PA}\gamma = \mathrm{BAC}.$

Thus AP, BP, CP bisect perpendicularly the sides of $\alpha\beta\gamma$, and P is the circum-centre of the same triangle.

[Thus any point is the circum-centre of the triangle whose vertices are the images of the isogonal conjugate point in the sides of the triangle of reference. For instance, the circum-centre of the triangle of reference is also the circum-centre of the triangle whose vertices are the images of the orthocentre in the sides.]

Now the hexagon $\mathrm{A}\gamma\mathrm{B}\alpha\mathrm{C}\beta$ = twice sum of triangles BQC, CQA, AQB $= 2\Delta\mathrm{ABC}$. But the same figure = twice sum of triangles $\mathrm{PA}\beta$, $\mathrm{PB}\gamma$, $\mathrm{PC}\alpha$. Also $\Delta\mathrm{PA}\beta : \Delta\mathrm{ABC} = \mathrm{AP} \,.\, \mathrm{A}\beta : \mathrm{AB} \,.\, \mathrm{AC} = \mathrm{AP} \,.\, \mathrm{AQ} : \mathrm{AB} \,.\, \mathrm{AC}$; whence the relation as stated. The applications (i.), (ii.), (iii.) are sufficiently obvious.



(II.) Draw the perpendiculars PL, PM, PN. Let the normal co-ordinates of P be (α, β, γ). Then the co-ordinates of Q will be $(\mathrm{K}/\alpha, \mathrm{K}/\beta, \mathrm{K}/\gamma)$, where

$$\mathrm{K} = 2\Delta/(a/\alpha + b/\beta + c/\gamma).$$

Now

$$\mathrm{AP} = \mathrm{MN}/\sin \mathrm{A} = \sqrt{(\beta^2 + \gamma^2 + 2\beta\gamma \cos \mathrm{A})}/\sin \mathrm{A};$$

therefore

$$\mathrm{AQ} = \mathrm{K}\sqrt{(\beta^2 + \gamma^2 + 2\beta\gamma \cos \mathrm{A})}/\beta\gamma \sin \mathrm{A}.$$

Therefore

$$\frac{\mathrm{AP} \,.\, \mathrm{AQ}}{\mathrm{AB} \,.\, \mathrm{AC}} = \frac{\mathrm{K}\,(\beta^2 + \gamma^2 + 2\beta\gamma \cos \mathrm{A})}{(\mathrm{AB} \,.\, \mathrm{AC} \sin \mathrm{A})\; \beta\gamma \sin \mathrm{A}} = \frac{\beta^2 + \gamma^2 + 2\beta\gamma \cos \mathrm{A}}{\beta\gamma \sin \mathrm{A}\,(a/\alpha + b/\beta + c/\gamma)};$$

therefore $$\Sigma \frac{\mathrm{AP} \,.\, \mathrm{AQ}}{\mathrm{AB} \,.\, \mathrm{AC}} = \frac{1}{a/\alpha + b/\beta + c/\gamma}\, \Sigma \frac{\beta^2 + \gamma^2 + 2\beta\gamma \cos \mathrm{A}}{\beta\gamma \sin \mathrm{A}}.$$

Now

$$\frac{\beta^2 + \gamma^2 + 2\beta\gamma \cos \mathrm{A}}{\beta\gamma \sin \mathrm{A}} = \frac{\mathrm{MN}^2}{\mathrm{PM} \,.\, \mathrm{PN} \sin \mathrm{A}} = \frac{\mathrm{AP} \,.\, \mathrm{MN}}{\mathrm{PM} \,.\, \mathrm{PN}} = \frac{\mathrm{AN} \,.\, \mathrm{PM} + \mathrm{AM} \,.\, \mathrm{PN}}{\mathrm{PN} \,.\, \mathrm{PM}}$$

$$= \frac{\mathrm{AN}}{\mathrm{PN}} + \frac{\mathrm{AM}}{\mathrm{PM}} = \frac{\mathrm{AN}}{\gamma} + \frac{\mathrm{AM}}{\beta};$$

therefore $$\Sigma \frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{\beta\gamma\sin A} = \frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma};$$

therefore $$\Sigma \frac{AP.AQ}{AB.AC} = 1.$$

(i.) In this case the median point and symmedian point are isogonal conjugates; therefore $(AG.AK)/(bc)+(BG.BK)/(ca)+(CG.CK)/(ab) = 1$; therefore $a.m_1.AK+b.m_2.BK+c.m_3.CK = \frac{2}{3}abc$.

(ii.) In this case, I is its own isogonal conjugate; therefore $AI^2/(bc)+BI^2/(ca)+CI^2/(ab) = 1$ or $a.AI^2+b.BI^2+c.CI^2 = abc$.

(iii.) In this case the orthocentre and circum-centre are isogonal conjugates; therefore

$$(AP.R)/(bc)+(BP.R)/(ca)+(CP.R)/(ab) = 1$$

or $$a.AP+b.BP+c.CP = abc/R.$$

5876. (T. Cotterill, M.A.)—(1) If (x_1, y_1), (x_2, y_2) are the perpendiculars from two conjugate foci of a conic upon any two of its conjugate lines x and y, prove that $(x_1y_2+y_1x_2)\sec(xy)$ is invariable. (2) Hence (or geometrically) show that conjugate foci of a conic touching CA, CB at A and B are foci of a conic touching AB and the reflexions of AB to CA and CB. (3) Prove that the same holds good for the sphere.

Solution by Professor Nanson.

(1) The lines p, α; q, β are conjugate with respect to $x^2/a^2+y^2/b^2 = 1$, if $a^2\cos\alpha\cos\beta+b^2\sin\alpha\sin\beta = pq$. Now

$$x_1, x_2 = p\pm ae\cos\alpha, \quad y_1, y_2 = q\pm ae\cos\beta;$$

therefore $x_1y_2+x_2y_1 = 2(pq-a^2e^2\cos\alpha\cos\beta) = 2b^2\cos(\alpha-\beta)$;

therefore $$(x_1y_2+x_2y_1)\sec(xy) = 2b^2.$$

(2) If F_1, F_2 are foci of a conic touching CA, CB at A, B, then F_1A, F_2A make equal angles with CA and are therefore equally inclined to AB and its reflexion to CA; and similarly for F_1B, F_2B. But these are the conditions that F_1, F_2 are the foci of a conic inscribed in the triangle formed by AB and its reflexions.

(3) For a sphero-conic we have $x^2+y^2+z^2 = 2k\alpha\beta$, where α, β are the cyclic planes. Hence two points xyz, $x'y'z'$ are conjugate if

$$xx'+yy'+zz' = k(\alpha\beta'+\alpha'\beta),$$

and, interpreted as in Salmon, § 245, this gives

$$(\xi_1\eta_2+\xi_2\eta_1)\sec(\xi\eta) = \text{const.},$$

where ξ, η are two conjugate points for a sphero-conic, and ξ_1, ξ_2; η_1, η_2 are the sines of their distances from the cyclic planes. Reciprocating, we get (1) where (x_1y_1), (x_2y_2) are the sines of the perpendiculars from two conjugate foci of a sphero-conic on two of its conjugate arcs x, y.

The proof in (2) applies also because the joins of the foci to any point of a sphero-conic make equal angles with the tangent at that point.

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